Lecture 13

Propulsion 3: Ideal performance of turbojets and turbofans

Outline
- Turbojet analysis - assumptions and goals
- Process summary
- State-by-state analysis
- Results
  - Thrust
  - Efficiency
  - Fuel consumption
- Effects of
  - Compressor pressure ratio
  - Flight Mach number
  - $\tau_3$ limit
- Sidebar topic: recuperation
**Outline (continued)**

- Afterburning turbojet
  - Analysis
  - Results - Thrust, Efficiency, Fuel consumption
  - Effects of
    - Compressor pressure ratio
    - Flight Mach number
    - \( \tau_{s,AB} \) limit

- Turbofan
  - Analysis
  - Results - Thrust, Efficiency, Fuel consumption
  - Effects of
    - Fan bypass ratio
    - Fan pressure ratio
    - Compressor pressure ratio
    - Flight Mach number
    - \( \tau_{s} \) limit

**Assumptions**

- Steady, quasi-1D
- Constant \( C_{p}, \gamma, P_{exit} = P_{ambient} \) (\( P_{9} = P_{1} \) in current notation)
- Isentropic except for heat addition process
- Heat addition at \( M << 1, FAR << 1 \) up to materials limit temperature \( T_{\lambda} \)

**Goals:** determine Specific Thrust, Thrust Specific Fuel Consumption and thermal & overall efficiency as a function of flight Mach number \( M_{1} \), turbine inlet temperature limit \( T_{\lambda} \), & compressor pressure ratio \( P_{3t}/P_{2t} \)

**Turbojet analysis**

**Assumptions**

- Steady, quasi-1D
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**Diagram:**

- \( \dot{m}_{a} \)
- \( \dot{m}_{f} \)
- \( \dot{m}_{a} + \dot{m}_{f} \)
- Diffuser
- Compressor
- Combustor
- Turbine
- Nozzle
Why use Brayton cycle to model gas turbines?

- Pressure compression ratio \( r = \) pressure expansion ratio (i.e., \( P_9 = P_1 \)), which yields maximum thrust (Lecture 11)
- Heat input at constant \( P \) realistic for steady-flow, \( M << 1 \) process (see Rayleigh flow analysis)
- Constant \( s \) compression/expansion corresponds to adiabatic and reversible processes - not totally true but not bad
- Notes on Brayton cycle \( P-v \) and \( T-s \) diagrams
  - \( v \) on \( P-v \) diagram is specific volume \( (v) \) (m\(^3\)/kg) which IS a property of the gas (can't use "cylinder volume" \( V \) as in unsteady-flow engines since there isn't a fixed mass with changing volume)
  - \( s \) is specific entropy \( (J/kg-K) \) which IS a property of the gas, heat transfer = \( \int Tds \) if mass doesn't change during heat addition
  - \( P-v \) diagrams not as useful as with unsteady-flow engines where we can use a cylinder pressure gauge to measure \( P \) vs. \( t \) and calculate \( V \) vs. \( t \) from crank angle to get \( P-V \) diagram for comparison with ideal cycle (would need a pressure gauge moving with the flow!)

Ideal turbojet cycle - process summary

<table>
<thead>
<tr>
<th>Process</th>
<th>Name</th>
<th>Const.</th>
<th>M</th>
<th>( P_1 )</th>
<th>( T_1 )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 → 2</td>
<td>Diffuser</td>
<td>s</td>
<td>↓ to 0</td>
<td>( P_2 = P_3 )</td>
<td>( T_2 = T_1 )</td>
<td>Decelerate incoming gas to ( M = 0 ); no work</td>
</tr>
<tr>
<td>2 → 3</td>
<td>Compression</td>
<td>s</td>
<td>0</td>
<td>( P_3/P_2 = \pi_c )</td>
<td>( T_3/T_2 = \pi_c ((\gamma - 1)/\gamma) )</td>
<td>Compressor work input = ( C_P(T_3 - T_2) )</td>
</tr>
<tr>
<td>3 → 4</td>
<td>Combustion (main combustor)</td>
<td>P</td>
<td>0</td>
<td>( P_4 = P_3 )</td>
<td>( T_4 = T_3 )</td>
<td>Add heat to ( T_4 = T_2/T_1 ) - turbine inlet temperature limit</td>
</tr>
<tr>
<td>4 → 5</td>
<td>Expansion through turbine (to pay for compressor work)</td>
<td>s</td>
<td>0</td>
<td>( P_5/P_4 = (T_4/T_3)^{\gamma/(\gamma-1)} )</td>
<td>( T_5 = T_4 - (T_3 - T_2) )</td>
<td>Turbine work output = ( C_P(T_4 - T_3) ) - compressor work input = ( C_P(T_3 - T_2) )</td>
</tr>
<tr>
<td>5 → 6</td>
<td>Expansion through turbine (to pay for fan work)</td>
<td>s</td>
<td>0</td>
<td>( P_6/P_5 = (T_5/T_4)^{\gamma/(\gamma-1)} )</td>
<td>( T_6 = T_5 - (T_4 - T_2) )</td>
<td>Turbine work output = ( C_P(T_5 - T_4) ) - fan work input = ( \alpha C_P(T_4 - T_2) )</td>
</tr>
<tr>
<td>6 → 7</td>
<td>Combustion (afterburner)</td>
<td>P</td>
<td>0</td>
<td>( P_7 = P_6 )</td>
<td>( T_7 = \tau_\lambda T_1 )</td>
<td>Add heat to ( \tau_\lambda AB = T_7/T_1 ) - afterburner temperature limit</td>
</tr>
<tr>
<td>7 → 9</td>
<td>Expansion through nozzle</td>
<td>s</td>
<td>↑</td>
<td>( P_9 = P_7 )</td>
<td>( T_9 = T_7 )</td>
<td>( P_s = P_1 )</td>
</tr>
</tbody>
</table>
Model shown is open cycle, where mixture is inhaled, compressed, burned, expanded then thrown away (not recycled).

In a closed cycle with a fixed (trapped) mass of gas to which heat is transferred to/from, 9 → 1 would be connected (Why don't we do this? Heat transfer is too slow!)

Compressor work = \( C_p(T_3-T_2) \) (matches compressor work)

Heat input = \( C_p(T_4-T_3) = C_p(T_3t-T_2t) \) since \( M_3 = M_4 = 0 \)

Turbine work = \( C_p(T_5-T_4) \)

KE out = \( C_p(T_5-T_3) \);

\( T_{3t} = T_{4t} \) since no heat or work

KE in = \( C_p(T_2-T_1) \);

\( T_{2t} = T_{1t} \) since no heat or work
Ideal turbojet cycle - analysis

Inlet conditions $M_1$, $T_1$, $P_1$: after diffuser (2), decelerate to $M_2 = 0$

$$M_2 = 0; \ T_2 = T_1 T_e; \ T_{st} = T_{st} = T_1 T_e; \ P_2 = P_1 T_e; \ P_{st} = P_{st}$$

$q_{1-2} = w_{1-2} = 0; \ \tau_r = 1 + \frac{y-1}{2} M_r^2 = "recovery\ temperature\ ratio"$

After compressor (3): isentropic compression by pressure ratio $\pi_c$

$$M_3 = 0; \ T_3 = T_{st} = T_{st} \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1} = T_1 T_e \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}; \ P_3 = P_{st} = P_{st} \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}$$

$q_{3-4} = 0; \ w_{2-3} = C_p(T_{st} - T_{st})$

After combustor (4): constant-pressure heat addition to $T_h$

$$M_4 = 0; \ T_4 = T_{st} = T_{st}; \ P_4 = P_{st} = P_{st} \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}$$

$q_{4-5} = C_p(T_{st} - T_{st}); \ w_{3-4} = 0$

After turbine (5): isentropic expansion to pay for compressor work

$$M_5 = 0; \ q_{4-5} = 0; \ w_{4-5} = w_{2-3} = -w_{2-3} = -C_p(T_{st} - T_{st}) = -C_p(T_{st} - T_{st})$$

$$T_5 = T_{st} = T_{st} = T_{st} = T_{st} - T_{st} \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1} = T_1 \left[1 - T_1 \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}\right]$$

$$P_5 = P_{st} = P_{st} \left(\frac{T_{st}}{T_4}\right)^{\gamma/\gamma-1} = P_1 \left(\frac{T_1}{\pi_1}\right)^{\gamma/\gamma-1} \left[1 - \left(\frac{T_1}{\pi_1}\right) \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}\right]^{\gamma/\gamma-1}$$

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Ideal turbojet cycle - analysis

After nozzle (9): isentropic expansion to $P_9 = P_{1} (P_e = P_a)$

$q_{4-5} = w_{4-5} = 0; \ P_0 = P_{st}; \ P_{w} = P_{st} = P_{st} \left(\frac{T_{st}}{T_{w}}\right)^{\gamma/\gamma-1} \left[1 - \left(\frac{T_{st}}{T_{w}}\right) \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}\right]^{\gamma/\gamma-1}$

$$P_{w}/P_0 = P_{st} \left(\frac{T_{st}}{T_{w}}\right)^{\gamma/\gamma-1} \left[1 - \left(\frac{T_{st}}{T_{w}}\right) \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}\right]^{\gamma/\gamma-1} / P_{st} = \left(1 + \frac{y-1}{2} M_5^2\right)^{\gamma/\gamma-1}$$

$$\Rightarrow M_5 = \sqrt{\frac{2}{y-1} \left(\frac{T_{st}}{T_{w}}\right)^{\gamma/\gamma-1} \left[1 - \left(\frac{T_{st}}{T_{w}}\right) \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}\right]^{\gamma/\gamma-1} - 1}$$

$$T_0 = T_{st} \left(P_{w}/P_0\right)^{\gamma/\gamma-1} = T_{st} \left[1 - T_{st} \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}\right]^{\gamma/\gamma-1} / \left[\left(\frac{T_{st}}{T_{w}}\right)^{\gamma/\gamma-1} \left[1 - \left(\frac{T_{st}}{T_{w}}\right) \left(\frac{\pi_c}{\pi_1}\right)^{\gamma/\gamma-1}\right]^{\gamma/\gamma-1}\right]$$

$$T_0 = T_{st} \left(T_{st} \left(T_{w}/T_{st}\right)^{\gamma/\gamma-1}\right)^{\gamma/\gamma-1}$$

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**Ideal turbojet cycle - Thrust**

- Recall (Lecture 11) 
  \[ \text{Thrust} = \dot{m}_a [(1 + \text{FAR})u_0 - u_t] + (P_0 - P_1)A_0 \]
  
  For \( \text{FAR} \ll 1, P_0 = P_1 \): 
  \[ \text{Thrust} \approx \dot{m}_a [u_0 - u_t] \]

  Specific Thrust (ST) is equivalent to: 
  \[ \frac{\text{Thrust}}{\dot{m}_a c_t} = \frac{u_0 c_0}{c_t} - \frac{u_t}{c_t} = M_0 \frac{\gamma R T_0}{\sqrt{\gamma R T_1}} - M_1 = \dot{m}_a \frac{T_0}{T_1} - M_1 \]

  
  \[ \text{ST} = \frac{2}{\gamma - 1} \left[ \frac{\tau_r (\pi_c)^{\gamma / (\gamma - 1)}}{1 - (\pi_c/\tau_r) \left( (\pi_c)^{\gamma / (\gamma - 1)} - 1 \right)} - 1 \right] \left[ \frac{\tau_r}{\tau_r - 1} - M_1 \right] \]

  
  \[ \text{ST} = \frac{2}{\gamma - 1} \left[ \tau_r \left( 1 - \frac{1}{\tau_r} \left( \frac{\pi_c}{\tau_r} \right)^{\gamma / (\gamma - 1)} \right) - \tau_r \left( \frac{\pi_c}{\tau_r} \right)^{\gamma / (\gamma - 1)} - 1 \right] - M_1 \]

- This gives the thrust in terms of:
  - Air flow (\( \dot{m}_a \))
  - Sound speed at ambient conditions (\( c_t \))
  - Flight Mach number \( M_1 \) and \( \tau_r = 1 + \left( \frac{\gamma - 1}{2} \right) M_1^2 \)
  - Compressor pressure ratio \( \pi_c \)
  - Materials-limited temperature at turbine inlet = \( \tau_r T_1 \)

**Ideal turbojet cycle - comments**

- Recall the assumptions required to get these results:
  - 1D steady flow
  - Ideal gas, constant specific heats
  - \( \text{FAR} \ll 1, P_0 = P_1 \)
  - Isentropic compression & expansion
  - Constant-P combustion at \( M_3 = M_4 = 0 \), add heat to \( \tau_r \) limit
  - Special case: \( \pi_c = 1 \) (ramjet, no compressor)

  \[ \text{Thrust} = \dot{m}_a c_t \left( \frac{2}{\gamma - 1} (\tau_r (1 - 1/\tau_r)) \right) - M_1 \left( \frac{\tau_r}{1 - \tau_r} \right) M_1 \text{ (no thrust at } M_1 = 0!) \]

  
  - If \( \tau_r = \tau_r (\pi_c)^{(\gamma - 1) / \gamma} \) then Thrust = 0
  - \( \tau_r \) materials limited temperature reached just by decelerating the gas to \( M = 0 \) and compressing it in the compressor
  - No “head room” in terms of temperature to enable heat addition
  - Could happen at unrealistically high \( \pi_c \) or (more realistically) at very high \( M_1 \) (thus high \( \tau_r \))

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Ideal turbojet cycle - notes on thrust

- Since either too low or too high πc leads to Thrust = 0, there is a value of πc that maximizes Thrust (but not any type of efficiency):

\[ \frac{\partial (\text{Thrust})}{\partial (\pi_c)} = 0 \Rightarrow \pi_c = \left( \frac{\sqrt{\tau_{\lambda}}}{\tau_r} \right)^{\gamma - 1} \]

- At this condition, \( T_3 = T_g = (\tau_{\lambda})^{\gamma/2}T_1 \), i.e. temperature at end of compression = temperature at end of expansion

- For typical \( \tau_{\lambda} = 5 \), \( \tau_r = 1.128 \) (M = 0.8), \( \gamma = 1.4 \), optimal \( \pi_c = 10.97 \)

Ideal turbojet cycle - thermal efficiency

Note \( T_i - T = T \left( 1 + \frac{\gamma - 1}{2} M^2 \right) - T = \frac{\gamma - 1}{2} M^2 T = \frac{\gamma - 1}{2} \frac{u^2}{c^2} \)

\[ \frac{\gamma - 1}{\gamma R} \frac{u^2}{2} = \frac{1}{C_p} \Rightarrow \text{KE per unit mass} = \frac{u^2}{2} = C_p(T_i - T) \] then

\[ \eta_h = \frac{\text{what you get}}{\text{what you pay for}} = \frac{\text{KE out} - \text{KE in}}{\text{KE in}} = \frac{u_i^2 / 2 - u_f^2 / 2}{C_p(T_i - T_f)} \]

\[ \frac{C_p(T_i - T_f) - C_p(T_f - T_i)}{C_p(T_i - T_f)} = T_i \frac{\tau_{\lambda} - \tau_r \left( \pi_c \right)^{\gamma/2} - 1}{\tau_r \left( \pi_c \right)^{\gamma/2} - T_i \tau_r + T_i} \]

\[ \frac{T_i \tau_{\lambda} - T_i \tau_r \left( \pi_c \right)^{\gamma/2}}{\tau_r \left( \pi_c \right)^{\gamma/2}} = 1 - \frac{1}{\tau_r \left( \pi_c \right)^{\gamma/2}} - \frac{1}{\tau_r \left( \pi_c \right)^{\gamma/2}} = 1 - \frac{1}{\tau_r \left( \pi_c \right)^{\gamma/2}} \]

\[ \eta_h = 1 - \frac{1}{\left( \pi_c \right)^{\gamma/2}} = 1 - \frac{1}{\left( r \right)^{\gamma/2}} \]

\[ r = \pi_c \pi_r \; \; \pi_r \; \text{"recovery pressure" } = \left( \tau_{\lambda} \right)^{\gamma/2} - 1 \]
**Ideal turbojet cycle - thermal efficiency**

- Note thermal efficiency \( \eta_{th} = 1 - \frac{1}{r^{(\gamma - 1)/\gamma}} \), where \( r = \frac{\pi_r}{\pi_c} \), i.e. the combined pressure rise due to “ram effect” compression (decelerating the gas from \( M_i \) to \( M = 0 \)) AND the mechanical compression - each has the same effect on thermal efficiency.

- Very similar to the Otto cycle (\( \eta_{th} = 1 - \frac{1}{r^{(\gamma - 1)/\gamma}} \)), where \( r \) is the volume (not pressure) ratio; why the difference? Otto is constant \( V \) heat addition and expansion back to the initial \( V \), whereas Brayton is constant \( P \) heat addition and expansion back to the initial \( P \).

- In either case \( \eta_{th} = 1 - \frac{T_{L}}{T_{H}} \) and \( T_{L}/T_{H} \) is the same for each Carnot strip in this cycle; \( T_{L}/T_{H} = (P_{L}/P_{H})^{(\gamma - 1)/\gamma} = (V_{L}/V_{H})^{(\gamma - 1)} \), thus \( \eta_{th} = 1 - (P_{L}/P_{H})^{(\gamma - 1)/\gamma} = 1 - (V_{H}/V_{L})^{(\gamma - 1)} \); the only difference is that Otto cycles are specified in terms of volume ratio whereas Brayton cycles are specified in terms of pressureratio.

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**Ideal turbojet cycle - fuel consumption**

- Recall (Lecture 11) Thrust Specific Fuel Consumption (TSFC):

\[
TSFC = \frac{\dot{m}_f}{\text{Thrust}} \frac{Q_R}{c_i^2} = \left( \frac{\dot{m}_a c_i}{\text{Thrust}} \right) \frac{\text{FAR} \cdot Q_R}{c_i^2} = \frac{\text{FAR} \cdot Q_R}{ST \gamma R T_i}
\]

- To compute TSFC is to compute FAR; energy balance on combustor (heat input = change in total enthalpy):

\[
\dot{m}_a Q_R = (\dot{m}_a + \dot{m}_f) C_p T_{4t} - (\dot{m}_a) C_p T_{3t} \rightarrow (\text{FAR})R Q_R = (1 + \text{FAR}) C_p T_{4t} - (1) C_p T_{3t}
\]

- But \( \text{FAR} < 1 \), \( T_{4t} = T_{L} \tau_\lambda \), \( T_{3t} = T_{L} \tau_\tau \left( \tau_{\pi} \right)^{\gamma - 1/\gamma} \) \( \Rightarrow \) \( (\text{FAR})R Q_R = C_p T_{4t} \tau_\lambda \tau_\tau \left( \tau_{\pi} \right)^{\gamma - 1/\gamma} \)

\[
\Rightarrow TSFC = \frac{1}{ST} \frac{(\text{FAR})R Q_R}{\gamma R T_i} = \frac{1}{ST} \frac{(\text{FAR})R Q_R}{C_p T_i} \gamma \frac{\tau_\lambda \tau_\tau \left( \tau_{\pi} \right)^{\gamma - 1/\gamma}}{\gamma R T_i} = \frac{\left( \tau_\lambda \tau_\tau \left( \tau_{\pi} \right)^{\gamma - 1/\gamma} \right)}{\gamma R T_i}
\]

\[
\Rightarrow TSFC = \frac{1}{ST \gamma - 1} \frac{\left( \tau_\lambda \tau_\tau \left( \tau_{\pi} \right)^{\gamma - 1/\gamma} \right)}{\gamma R T_i}
\]

Ideal turbojet; use ST from page 11
Note on FAR: how do we know that we can add enough fuel to reach the $\tau_\lambda$ limit before we run out of $O_2$ in the air? From the previous page, energy balance on combustor:

$$\text{FAR} \frac{Q_R}{C_P T_1} = \tau_\lambda - \tau_r \left( \frac{\pi_c}{\gamma} \right)^{\gamma-1/\gamma}$$

Using realistic numbers $\text{FAR}_{\text{stoich}} = 0.068$, $Q_R = 4.3 \times 10^7$ J/kg, $C_p = 1400$ J/kgK, $T_1 = 300K$, $(\text{FAR}_{\text{stoich}} Q_R)/(C_P T_1) = 7.2$, and we require $0 < \text{FAR} < \text{FAR}_{\text{stoich}}$, thus at stoichiometric

$$\tau_\lambda - \tau_r \left( \frac{\pi_c}{\gamma} \right)^{\gamma-1/\gamma} = 7.3$$

But typically the maximum allowable turbine inlet temperature $T_{4t}$ is at most 1800K; with $T_1 = 300K$, $\tau_\lambda = 6 < 7.2$, so we can never add the stoichiometric amount of fuel - we reach the materials limit first.

Also note that FAR can be calculated via

$$(\text{FAR}) \frac{Q_R}{C_P T_1} = \tau_\lambda - \tau_r \left( \frac{\pi_c}{\gamma} \right)^{\gamma-1/\gamma} \Rightarrow \text{FAR} = \left( \frac{C_P T_1}{Q_R} \right) \left( \tau_\lambda - \tau_r \left( \frac{\pi_c}{\gamma} \right)^{\gamma-1/\gamma} \right)^{\gamma-1/\gamma}$$

For typical values $\pi_c = 30$, $\gamma = 1.4$, $\tau_\lambda = 5$, $\tau_r = 1$ ($M_1 = 0$), $Q_R = 4.3 \times 10^7$ J/kgK, $C_p = 1400$ J/kgK, $T_1 = 300K$:

$$\text{FAR} \approx 0.023; \phi = \text{FAR/} \text{FAR}_{\text{stoich}} = 0.34 << 1$$

Wait - isn't $\phi = 0.34$ too lean to burn? For premixed hydrocarbon-air flame, yes, for non-premixed flame (e.g. spray flame, like diesel but continuous, not injected at discrete times), not a problem.

Also recall $\eta_{\text{overall}} = M_1/\text{TSFC}$
Ideal turbojet cycle - results - effect of $\pi_c$

- Next few slides: baseline values $M_1 = 0.8$, $\gamma = 1.4$, $\tau_\lambda = 5$, $\pi_c = 30$
- One parameter changed, others fixed
- For very low $\pi_c$, $\eta_{th}$ is low, so both thrust and TSFC are low
- At very (unrealistically) high $\pi_c$, very little fuel can be added, thus thrust decreases, but TSFC is great!

$$\begin{align*}
M_1 &= 0.8 \\
\gamma &= 1.4 \\
\tau_\lambda &= 5 \\
\pi_c &= \text{varies}
\end{align*}$$

Double-click plot to open Excel spreadsheet

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Ideal turbojet cycle - results - effect of $M_1$

- ST decreases as $M_1$ increases: less fuel can be added ($\tau_\lambda$ limit)
- $\eta_{th}$ increases as $M_1$ increases: total P ratio = $\pi_r \pi_c$ increases
- $\eta_{prop} = 2(u_1/u_9)/(1+ u_1/u_9)$ increases as $M_1$ increases: $u_1/u_9 \to 1$
- As a result, $\eta_{overall} = \eta_{th}\eta_{prop}$ increases as $M_1$ increases
- TSFC increases as $M_1$ increases since $\text{TSFC} = M_1/\eta_{overall}$

$M_1 = \text{varies}$
$\gamma = 1.4$
$\tau_\lambda = 5$
$\pi_c = 30$

T-s diagrams similar to $\pi_c$ effect - tall skinny diagrams for high $M_1$, “banana” shaped cycles for low $M_1$, and “fat” cycles for intermediate $M_1$
Ideal turbojet cycle - results - effect of $\tau_{\lambda}$

- As $\tau_{\lambda}$ increases, specific thrust $ST$ increases but so does TSFC due to lower propulsive efficiency ($\eta_{prop} = 2(u_1/u_9)/(1 + u_1/u_9)$; for fixed $u_1$, $u_9$ increases with increasing $\tau_{\lambda}$).

- At very low $\tau_{\lambda}$, no heat addition is possible, thus no thrust.

\[ M_s = 0.8 \quad \gamma = 1.4 \quad \tau_{\lambda} \text{ varies} \quad \pi_c = 30 \]

T-s diagrams just show increasing heat addition as $\tau_{\lambda}$ increases, no change in $\eta_{th}$ (which can be seen on T-s) but decreases in $\eta_{prop}$ (which can't be seen on T-s or P-v)
Sidebar topic: Recuperation

- If $T_g > T_3$, some heat from exhaust can be transferred to incoming air after compression but before combustion to reduce fuel consumption — recuperation
- Only practical for stationary power generation ($M_1 = 0$) because heat exchangers are too large and heavy for aircraft engines
- For $M_1 = 0$, $T_g > T_3$ requires $\pi_c < \tau_\lambda^{(M=0)}$; else no recuperation possible
- Analysis same as baseline Brayton except for calculation of heat addition

Recuperation - analysis

$$M = 0: \ T_2 = T_1; \ T_3 = T_2 \pi_c^{^{(M=0)}} = T_4 R; \ T_4 = \tau_\lambda T_1; \ T_9 = T_{\,\lambda} T_1 / R; \ R = \pi_c^{^{(M=0)}}$$

$$\eta_{th} = \frac{\text{Net work}}{\text{Net heat input}} = \frac{C_p(T_4 - T_9) - C_p(T_3 - T_2)}{C_p(T_4 - T_3) - \epsilon C_p(T_9 - T_1)}$$

$$\epsilon = \text{Heat exchanger effectiveness} = \frac{\text{Actual heat transfer}}{\text{Maximum possible heat transfer}}$$

$$\eta_{th} = \frac{(\tau_\lambda T_1 - \tau_\lambda T_3 / R) - (T_4 R - T_1)}{(\tau_\lambda T_1 - T_1 R) - \epsilon(T_4 R - T_1)} = \frac{(\tau_\lambda - R)(R - 1)}{\tau_\lambda (R - \epsilon) - R^2(1 - \epsilon)}; \ R = \pi_c^{^{(M=0)}}$$

Checks: $\epsilon = 0$ (no recuperation), $\eta_{th} = 1 - 1 / R = 1 - 1 / r$, (standard Brayton cycle)

$\epsilon = 1, R \to 1$ (perfect recuperation, no $T$ rise due to compression):

$$\eta_{th} \to 1 - 1 / \tau_\lambda = 1 - 1 / (T_4 / T_1) = 1 - (T_4 / T_1)$$ (Same as Carnot since heat addition only at $T = T_4$ and heat rejection only at $T = T_1$)

$R \to 1$ (no compression): $\eta_{th} \to 0$ (unless $\epsilon = 1$)
Recuperation - results

- Substantially increases efficiency when $\pi_c$ is low
- No recuperation possible at high $\pi_c$ ($\approx 32$ for case shown) because then $T_9 < T_3$ (exhaust $T < $ post-compression $T$)
- High exchanger efficiency required for best results
- At sufficiently high $\varepsilon$ ($\approx 0.52$ for case shown), there is an optimum $\pi_c$ that maximizes efficiency (optimum $\pi_c \to 1$ as $\varepsilon \to 1$)

\[
\eta = \eta_{\text{Carnot}} = 1 - 1/\tau_\lambda = 0.8 \quad \tau_\lambda = 5
\]

Afterburning turbojet cycle

- In turbojet cycle FAR $< \text{FAR}_{\text{stoich}}$ since $\tau_\lambda$ limit well below $T_f$ for stoichiometric HC-air, thus much $O_2$ goes never mixes with fuel & burns - lost opportunity to generate power/thrust
- Solution: afterburner - add more fuel AFTER turbine
  - No moving parts after turbine, easier to cool $\Rightarrow \tau_{\lambda,\text{AB}} > \tau_\lambda$

\[
\text{P-v diagram}
\]

\[
\text{T-s diagram}
\]
**Afterburning turbojet cycle - analysis**

- So why doesn't everyone use an afterburner?
  - Thermal efficiency is low! vs.
  - Also - not useful to generate shaft work - can't put 2nd turbine after afterburner - too hot!
  - Analysis of afterburning turbojet same as non-afterburning up to state 5 (after turbine); from page 9

\[
M_s = 0; \quad q_{4 \rightarrow 5} = 0; \quad w_{4 \rightarrow 5} = C_p(T_{4t} - T_{5t}) = -C_p(T_{2t} - T_{3t})
\]

\[
T_5 = T_{5t} = T_{3t} + (T_{2t} - T_{3t}) = T_{3t} \left[ \tau_r - \pi_{\frac{\gamma - 1}{\gamma}} \right] = T_{3t} \left[ \tau_r - \pi_{\frac{\gamma - 1}{\gamma}} - 1 \right]
\]

\[
P_5 = P_{5t} = P_{4t} \left( \frac{T_{5t}}{T_{4t}} \right)^{\gamma - 1} = P_T \left( \frac{T_r}{\pi_{\frac{\gamma - 1}{\gamma}}} \right)^{\gamma - 1} \pi_{\frac{\gamma - 1}{\gamma}} \left[ 1 - \left( \frac{T_r}{\pi_{\frac{\gamma - 1}{\gamma}}} \right)^{\gamma - 1} \right]^{\gamma - 1}
\]
Ideal afterburning turbojet cycle - analysis

- After afterburner (7): constant-P heat addition to $T_9 = T_{1r_{A,B}}$
  
  $M_q = 0$; $T_q = T_{1r_{A,B}}$; $q_{s_{A,B}} = C_p (T_{7t} - T_{5s})$; $w_{s_{A,B}} = 0$

  $$P_7 = P_{7t} = P_1 \left( \frac{\gamma}{\gamma-1} \pi_c \left[ 1 - \left( \frac{\tau_r}{\tau_x} \right) \left( \pi_x \right)^{\gamma/\gamma-1} \right] \right)^{\gamma-1}$$

- After nozzle (9): isentropic expansion to $P_9 = P_1$

  $$q_{t_{9}} = w_{t_{9}} = 0; \quad P_9 = P_{1}; \quad P_{w_9} = P_{5t} = P_1 \left( \frac{\gamma}{\gamma-1} \pi_c \left[ 1 - \left( \frac{\tau_r}{\tau_x} \right) \left( \pi_x \right)^{\gamma/\gamma-1} \right] \right)^{\gamma-1}$$

  $$\left( \frac{P_{w_9}}{P_9} \right) = P_1 \left( \frac{\gamma}{\gamma-1} \pi_c \left[ 1 - \left( \frac{\tau_r}{\tau_x} \right) \left( \pi_x \right)^{\gamma/\gamma-1} \right] \right)^{\gamma-1}$$

  $$= M_q = \sqrt[\gamma-1]{\frac{\gamma}{\gamma-1} \pi_c \left[ 1 - \left( \frac{\tau_r}{\tau_x} \right) \left( \pi_x \right)^{\gamma/\gamma-1} \right] - 1}$$

  - Since all pressure ratios are same, $M_q$ is same as without afterburner!

Ideal afterburning turbojet cycle - analysis

- ... but more heat has been added, so $T_9$ is larger, so $c_g = \sqrt{\gamma RT_9}$ and thus $u_9$ is larger

  $T_{9t} = T_{9t_{A,B}}$; $T_9 = T_{1r_{A,B}} (P_1/P_3)^{\gamma/\gamma-1}$

  $$T_9 = T_{1r_{A,B}} \left[ \left( \pi_x \right)^{\gamma/\gamma-1} \left[ 1 - \left( \frac{\tau_r}{\tau_x} \right) \left( \pi_x \right)^{\gamma/\gamma-1} \right] \right]$$

  Specific Thrust (ST) = $\frac{\text{Thrust}}{m_{c_1}} = \frac{T_9}{T_1} = M_1$

  $$ST = \frac{2}{\gamma-1} \pi_c \left[ \left( \pi_x \right)^{\gamma/\gamma-1} \left[ 1 - \left( \frac{\tau_r}{\tau_x} \right) \left( \pi_x \right)^{\gamma/\gamma-1} \right] \right] - M_1$$

  $$\left[ \frac{2}{\gamma-1} \pi_c \left[ \left( \pi_x \right)^{\gamma/\gamma-1} \left[ 1 - \left( \frac{\tau_r}{\tau_x} \right) \left( \pi_x \right)^{\gamma/\gamma-1} \right] \right] - M_1 \right] - M_1$$

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Afterburning turbojet cycle - thrust

- $\pi_c = 1$ (ramjet, no compressor)

\[
\text{Thrust} = \frac{\dot{m}_c c_1}{\gamma - 1} \left[ \tau_{\lambda,AB}(1 - 1/\tau_r) \right] - M_i \left( \frac{\tau_{\lambda,AB}}{\tau_r} - 1 \right) M_i \text{ (same as turbojet)}
\]

(i.e. an afterburning ramjet is just a ramjet since no compressor)

- Too high $\pi_c$ leads to net work = 0 for main part, thus pressure ratio = 1 for afterburner part, thus no work for either part
- Too low $\pi_c$ leads to no pressure ratio for main or afterburner part, thus no work for either part

- Value of $\pi_c$ at maximum Thrust:

\[
\frac{\partial}{\partial \pi_c} \left( \frac{\dot{m}_c c_1}{\gamma - 1} \right) = 0 \Rightarrow \pi_c = \left( \frac{\tau_{\lambda} + \tau_r}{2\tau_r} \right)^{\gamma-1}
\]

for which $T_{3t} = T_{5t}$!

( Doesn't depend on $\tau_{\lambda,AB}$ at all!)

Afterburner - thermal efficiency

\[
\eta_{th} = \frac{\text{KE out} - \text{KE in}}{\text{heat in}} = \frac{u_{2t}^2/2 - u_{2t}^2/2}{C_p(T_{4t} - T_{3t}) + C_p(T_{7t} - T_{3t})} = \frac{C_p(T_{4t} - T_{5t}) - C_p(T_{7t} - T_{5t})}{C_p(T_{4t} - T_{5t}) + C_p(T_{7t} - T_{5t})}
\]

\[
T_{3t} \tau_{\lambda,AB} = T_i \tau_{\lambda,AB} \left[ \frac{\tau_{\lambda}^\gamma}{\gamma} \left[ 1 - \left( \frac{\tau_{\lambda}}{\tau_r} \right) \left( \frac{\tau_{\lambda}}{\gamma/\gamma - 1} \right) \right] \right]^{-1} - T_i \tau_r + T_i
\]

\[
= \left( T_{3t} \tau_{\lambda,AB} - T_i \tau_{\lambda,AB} \left[ \frac{\tau_{\lambda}^\gamma}{\gamma} \left[ 1 - \left( \frac{\tau_{\lambda}}{\tau_r} \right) \left( \frac{\tau_{\lambda}}{\gamma/\gamma - 1} \right) \right] \right] \right)
\]

As expected, $\eta_{th}$ is lower; similar to Diesel (heat addition at lower $P$ (thus lower $T$) than “baseline” turbojet (or Otto cycle)
To compute TSFC we need FAR; now 2 fuel streams, again do energy balance on combustors:

\[
m_f Q_R = m_a C_p (T_{4t} - T_{3t}) + m_a C_p (T_{7t} - T_{5t})
\]

\[
(FAR) Q_R = C_p (T_{4t} - T_{3t}) + C_p (T_{7t} - T_{5t})
\]

but \( T_{4t} = T_{1t} \tau_{\lambda}, \ T_{3t} = T_{1t} \tau_{r} / (\gamma -1), \ T_{7t} = T_{1t} \tau_{\lambda,AB}, \ T_{5t} = T_{1t} \tau_{r} / (\gamma -1), \)

\[
T_{5t} = T_{1t} \left[ \frac{\tau_{\lambda} - \tau_{r}}{(\gamma -1)} \right]^{\gamma -1}
\]

\[
\Rightarrow (FAR) Q_R = C_p T_{1t} \tau_{\lambda,AB} - \tau_{r}
\]

\[
TSFC = \left( \frac{\text{Thrust}}{m_a c_i} \right)^{\gamma -1} \frac{FAR \cdot Q_R}{C_p T_{1t}} \frac{1}{\gamma -1} \text{ as with turbojet}
\]

\[
\Rightarrow TSFC = \frac{1}{ST} \frac{1}{\gamma -1} \left( \frac{\tau_{\lambda,AB} - \tau_{r}}{\gamma -1} \right)
\]

ST from page 32

Afterburning turbojet - effect of \( \pi_c \)

- Maximum ST increases 1.5x, but \( \eta_{\text{overall}} \) decreases substantially
- For low \( \pi_c \), \( \eta_{\text{th}} \) is low, so both thrust and TSFC are low
- At high \( \pi_c \), “check mark” cycle - main cycle has high \( \eta_{\text{th}} \), but low work, afterburning cycle is banana-like - low \( \eta_{\text{th}} \), thus \( \eta_{\text{overall}} \) is low

Standard turbojet from page 19
Afterburning turbojet - effect of $\pi_c$

- High $\pi_c$: T-s diagrams show “check mark” cycle - tall skinny main T-s diagrams & “banana” shaped afterburner cycles
- Low $\pi_c$: cycle is banana-shaped; fattest cycles for intermediate $\pi_c$

$M_1 = 0.8$
$\pi_c$ varies
$\tau_\lambda = 5$
$\tau_{\lambda,AB} = 7$

Afterburning turbojet - effect of $M_1$

- High $M_1$: ST decreases since less fuel can be added ($\tau_\lambda$ limit)
- $\eta_{th}$ increases with $M_1$ since total pressure ratio = $\pi_r \pi_c$ increases
- $\eta_{prop} = 2(u_1/u_9)/(1+ u_1/u_9)$ increases with $M_1$ since $u_1/u_9 \to 1$
- TSFC increases even though $\eta_{overall} = \eta_{th}\eta_{prop}$ increases since PDR’s definition of TSFC has $M_1$ in it – biased against high $M_1$

$M_1 = \text{varies}$
$\pi_c = 10.97$
$\tau_\lambda = 5$
$\tau_{\lambda,AB} = 7$
Afterburning turbojet cycle - results - effect of $M_1$

- T-s diagrams similar to $\pi_c$ effect - “banana” cycles for low $\pi_c$, “fat” cycles for intermediate $\pi_c$

<table>
<thead>
<tr>
<th>M_1 = varies</th>
<th>$\pi_c = 10.97$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_\lambda = 5$</td>
<td>$\tau_{\lambda,AB} = 7$</td>
</tr>
</tbody>
</table>

$\pi_c = 10.97$

Afterburning turbojet - effect of $\tau_{\lambda,AB}$

- As $\tau_{\lambda,AB}$ increases, Thrust increases but so does TSFC due to lower $\eta_{prop} = 2(u_1/u_9)/(1+ u_1/u_9)$; $u_9$ increases for fixed $u_1$
- At very low $\tau_{\lambda,AB}$, no afterburning heat addition is possible, thus cycle reverts to plain turbojet

$M_1 = 0.8$

<table>
<thead>
<tr>
<th>$\pi_c = 33.03$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_\lambda = 5$</td>
</tr>
<tr>
<td>$\tau_{\lambda,AB} = \text{varies}$</td>
</tr>
</tbody>
</table>
**Afterburning turbojet - effect of $\tau_{\lambda,AB}$**

- No change in $\eta_{lm}$ of afterburning part as $\tau_{\lambda,AB}$ increases (which can be seen on T-s) but decrease in $\eta_{prop}$ (which can't be seen on T-s or P-v) due to higher $u_9$ for fixed $u_1$

\[ M_1 = 0.8 \]
\[ \pi_c = 33.03 \]
\[ \tau_{\theta} = 5 \]
\[ \tau_{\lambda,AB} = \text{varies} \]

**Turbofan**

- Turbojets and afterburning turbojets 'suffer' from large exhaust velocity $u_9$ - provides large thrust per unit mass flow (specific thrust, ST) but poor propulsive efficiency, thus poor TSFC and overall efficiency
- Lecture 11: what you really want to do is accelerate $\infty$ mass of air by $1/\infty$ amount, thus $u_9 = u_1 (1 + 1/\infty)$, thus propulsive efficiency $\to 1$
- How to improve propulsive efficiency? **Turbofan**
  - Extract more power from turbine than needed to drive compressor
  - Use extra power to drive “fan” that accelerates large mass of air by $1$/large amount
  - No free lunch, combusted stream produces less thrust
Turbofan - P-v & T-s diagrams

- P-v and T-s diagrams look exactly the same as turbojet since an isentropic expansion through a nozzle or turbine look the same.

P-V & T-s diagrams for turbofan

- Turbine work for compressor = \( C_p(T_4 - T_3) = C_p(T_4^t - T_3^t) \)
- Turbine work for fan = \( C_p(T_5 - T_6) = C_p(T_5^t - T_6^t) \)
- KE out = \( C_p(T_8^t - T_9^t) \); \( T_8^t = T_9^t \)
- KE in = \( C_p(T_2^t - T_3^t) \)

\[ P = \text{constant} \]

\[ T = \text{constant} \]
Turbofan - analysis

- So why doesn’t everyone use a fan? (They do now)
- But fan is large, so low specific thrust if you include ALL air flow
- Analysis of turbofan same as non-afterburning up to state 5 (after turbine for compressor work); from page 9:

\[ M_3 = 0; \quad q_4 \rightarrow q_5 = 0; \quad w_4 \rightarrow w_5 = C_\rho(T_{4i} - T_{5i}) = -C_\rho (T_{2i} - T_{3i}) \]

\[ T_3 = T_{5i} = T_{4i} + (T_{2i} - T_{3i}) = T_i \tau + T_i \tau - T_i (\pi_c) \frac{\gamma - 1}{\gamma} = T_i \left[ \tau - \tau_i (\pi_c) \frac{\gamma - 1}{\gamma} - 1 \right] \]

\[ P_5 = P_{5i} = P_{4i}(T_{5i}/T_{4i})^{\gamma - 1} = P_i(\tau_i) \frac{\gamma - 1}{\gamma} \right) \frac{\gamma - 1}{\gamma} \]

\[ \text{Fan: use prime (’) superscript for (non-combusting) fan stream:} \]

\[ M_4' = 0; \quad T_4' = T_i \tau_i; \quad T_5' = T_i \tau_i; \quad P_4' = P_i(\tau_i) \frac{\gamma - 1}{\gamma} = T_i \tau_i (\pi_c) \frac{\gamma - 1}{\gamma} \]

\[ q_{2 \rightarrow 3} = 0; \quad w_{2 \rightarrow 3} = C_\rho(T_{2i} - T_{3i}) \]

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Turbofan - analysis

- Fan stream continued…

\[ q_3 \rightarrow q_4 = w_3 \rightarrow w_4 = 0; \quad P_4' = P_i; \quad P_5' = P_i(\tau_i) \frac{\gamma - 1}{\gamma} \pi_c \]

\[ P_{5i}'/P_{4i}' = P_i(\tau_i) \frac{\gamma - 1}{\gamma} \pi_c \frac{1}{P_i} \left( 1 + \frac{\gamma - 1}{2} (M_\rho) \right) \]

\[ \Rightarrow M_\rho = \frac{2}{\frac{\gamma - 1}{2} (\pi_c) \frac{\gamma - 1}{\gamma} - 1} \]

\[ T_5' = T_i \left( P_{5i}'/P_{4i}' \right) \frac{\gamma - 1}{\gamma} = T_i \tau_i \left( P_i \pi_c \right) \frac{\gamma - 1}{\gamma} \left( P_i \pi_c \right) \frac{\gamma - 1}{\gamma} \]

\[ T_6' = T_i \tau_i \left( \frac{\gamma - 1}{\gamma} \left( P_i \pi_c \right) \frac{\gamma - 1}{\gamma} \right) \frac{\gamma - 1}{\gamma} = T_i \]

Note fan stream exit temperature \( T_6' \) is same as ambient temperature \( T_1 \) - gas undergoes reversible adiabatic compression then expansion back to initial pressure (but fan work input comes out as KE of fan stream)

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Turbofan - analysis

- Need to complete the energy balance on the main stream (fan work = turbine work) to determine properties after fan turbine
- Define Bypass Ratio \( \alpha \) = ratio of mass flow through fan to mass flow through compressor \( \alpha = \dot{m}_f / \dot{m}_c \)

\[
M_\alpha = 0; \quad q_{i\rightarrow o} = 0; \quad w_{s\rightarrow o} = \dot{m}_c C_p (T_{i\rightarrow o} - T_{o\rightarrow o}) = -\dot{m}_f C_p (T_{i\rightarrow o} - T_{o\rightarrow o})
\]

\[
\Rightarrow T_{i\rightarrow o} = T_{o\rightarrow o} - \alpha (T_{i\rightarrow o} - T_{o\rightarrow o}) = T_i \left[ \tau_c - \tau_f \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right) \right] - \alpha \left[ \frac{T_c \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right)}{\tau_c - \tau_f \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right)} \right]^{\gamma / \gamma - 1}
\]

\[
P_{o\rightarrow o} = P_o = \frac{P_{i\rightarrow o}}{\tau_c \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right)^{\gamma / \gamma - 1}}
\]

Turbofan - analysis

- Expand isentropically back to \( P_9 = P_1 \)

\[
P_{9\rightarrow 1} = P_{1\rightarrow 1} \left( \left( \frac{T_9}{T_1} \right)^{\gamma - 1} \right) \left( 1 - \left( \frac{T_9}{T_1} \right) \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right) \right) = \frac{1 + \frac{M_9^2}{2} - 1}{\gamma - 1} = \frac{2}{\gamma - 1} \left( \frac{T_9}{T_1} \right)^{\gamma / \gamma - 1} \left( 1 - \left( \frac{T_9}{T_1} \right) \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right) \right)\]

\[
P_9 = P_1 \frac{P_{9\rightarrow 1}}{\gamma - 1} = \frac{1 + \frac{M_9^2}{2} - 1}{\gamma - 1} = \frac{2}{\gamma - 1} \left( \frac{T_9}{T_1} \right)^{\gamma / \gamma - 1} \left( 1 - \left( \frac{T_9}{T_1} \right) \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right) \right)
\]

\[
M_9 = \frac{2}{\gamma - 1} \left( \frac{T_9}{T_1} \right) \left( \left( \frac{\gamma}{\gamma - 1} \right)^{\gamma / \gamma - 1} - 1 \right)\]

- \( T_9 \) is the same with or without fan since in either case you’re expanding isentropically back to \( P_9 = P_1 \) (but of course \( M_9 \) and thus \( T_{9t} \) are different; exit velocity \( u_9 \) is less with fan due to the energy extracted from the main stream to pay for the fan work)

\[
T_9 = T_1 \frac{T_9}{T_1} \left( \frac{T_9}{T_1} \right)^{\gamma / \gamma - 1}
\]
**Turbofan - analysis - Thrust**

Then finally compute the Thrust (assuming \( P_9 = P_1 \), FAR << 1)

\[
\text{Thrust} = \dot{m}_a \left( (1 + \text{FAR})u_0 - u_1 \right) + (P_9 - P_1)A_0 + \dot{m}_a \left( u_9' - u_1 \right)
\]

\[
\text{Thrust} = \dot{m}_a [u_0 - u_1] + \dot{m}_a [u_9' - u_1]
\]

\[
\frac{\text{Thrust}}{(\dot{m}_a + \dot{m}_a')c_i} = \frac{\dot{m}_a [u_0 - u_1]}{(\dot{m}_a + \dot{m}_a')c_i} + \frac{\dot{m}_a [u_9 - u_1]}{(\dot{m}_a + \dot{m}_a')c_i}
\]

\[
\frac{\text{Thrust}}{(\dot{m}_a + \dot{m}_a')c_i} = \frac{1}{1 + \alpha} \left[ M_9 \sqrt{\frac{T_0}{T_1}} - M_1 \right] + \alpha \left[ M_9 \sqrt{\frac{T_0}{T_1}} - M_1 \right]
\]

\[
\frac{\text{Thrust}}{(\dot{m}_a + \dot{m}_a')c_i} = \frac{1}{1 + \alpha} \left[ M_9 \sqrt{\frac{T_0}{T_1}} - M_1 \right] + \alpha \left[ \frac{2}{\gamma - 1} \left( \tau (\pi_c) \right)^{\gamma - 1} - 1 \right] \left[ T_i - M_1 \right]
\]

\[
\frac{\text{Thrust}}{(\dot{m}_a + \dot{m}_a')c_i} = \frac{1}{1 + \alpha} \left[ M_9 \sqrt{\frac{T_0}{T_1}} - M_1 \right] + \alpha \left[ \frac{2}{\gamma - 1} \left( \tau (\pi_c) \right)^{\gamma - 1} - 1 \right] - M_1
\]

Where \( M_9 \) (for main stream) is given on the previous page

**Turbofan - analysis - TSFC**

- Now 3 parameters: \( \alpha, \pi_c, \pi_c' \); not just \( \pi_c \) as with the turbojet
- TSFC calculation is almost same as with turbojet, since same amount of fuel is used, but thrust is different of course:

\[
\text{TSFC} = \frac{\dot{m}_a Q_r}{\text{Thrust} \cdot c_i} = \frac{\dot{m}_a (\dot{m}_a + \dot{m}_a')c_i \cdot \text{FAR} \cdot Q_r}{\text{Thrust} \cdot c_i}
\]

\[
= \frac{1}{1 + \alpha} \left( \frac{\text{Thrust}}{(\dot{m}_a + \dot{m}_a')c_i} \right)^{-1} C_p \left( \frac{T_0}{T_1} - T_0 \right) = \frac{1}{1 + \alpha} \gamma \frac{1}{1 + \alpha} \frac{1}{\gamma - 1} \left( \tau (\pi_c) \right)^{\gamma - 1}
\]

- “It can be shown” (with A LOT of algebra) that a value of \( \alpha \) minimizes TSFC:

\[
\frac{\partial (\text{TFSC})}{\partial (\alpha)} = 0 \Rightarrow 2(u_0 - u_1) = (u_9 - u_1)
\]

At this optimum, thrust per unit mass of the main stream (\( u_0 - u_1 \)) is half that of the fan stream (\( u_9 - u_1 \)), i.e., the main stream is bled almost completely out of KE to feed the greedy fan (by drawing more power out of the turbine)
Turbofan - analysis – TSFC & $\eta_{th}$

- The optimal value of $\alpha$ (at $\partial$(TFSC)/$\partial(\alpha) = 0$) is

$$\alpha_{\text{optimal}} = \frac{1}{\left(\frac{T_1}{T_\tau}\right)^{\gamma-1}} \left[ 1 - \frac{1}{\left(\frac{T_1}{T_\tau}\right)^{\gamma-1}} \left( \frac{1}{4} \left( \frac{T_1}{T_\tau} \right)^{\gamma-1} - 1 + \sqrt{\frac{T_1}{T_\tau} - 1} \right) \right]$$

- Thermal efficiency $\eta_{th}$ of turbofan is exactly same as turbojet - same T-s cycle, just different “stopover” points 5 & 6 along the T-s path - the advantage of the turbofan is the propulsive efficiency $\eta_{prop}$ (which you can’t see on T-s diagram), not the thermal efficiency (which does show on T-s)

Turbofan - effect of fan bypass ratio $\alpha$

- Specific thrust decreases considerably as $\alpha$ increases (more mass of air being accelerated by a small amount)
- TSFC decreases, $\eta_o$ increases as $\alpha$ increases, but little benefit beyond a point ($\eta_{prop}$ already good; recall $\eta_o = \eta_{th}\eta_{prop}$; fan provides NO BENEFIT to $\eta_{th}$)
- As promised, optimum $\alpha$ (≈ 5 in this case); improves $\eta_o$ ≈ 50%
There is a limit to the value of the fan – for large $\alpha$, fan area is large, increases the drag \( D = C_D \rho_1 u_1^2 A_1 / 2 \), \( C_D \) = drag coefficient; \( A_1 \) = area of inlet (main + fan streams)

- D must be subtracted from Thrust

\[ Drag = C_D \rho_1 u_1^2 A_1 / 2 = C_d (\rho_1 u_1 A_1) u_1 / 2 = C_d (1 + \alpha) \dot{m}_a u_1 / 2 = C_d (1 + \alpha) \dot{m}_a c \tau \lambda M_1 / 2 \]

- Optimal $\alpha \approx 2$ with large drag ($C_D = 0.7$) compared to optimal $\alpha \approx 5$ with no drag

- Little effect of fan pressure ratio $\pi_c'$ except at low ratios, where we revert to turbojet, and high ratios where we bleed too much energy from main stream (recall optimum is $u_2' - u_1 = 2(u_2 - u_1)$)

- Specific thrust small because $\alpha$ fixed, so even at $\pi_c' = 1$ (no fan), mass of air going through fan is included in ST calculation
Turbofan - effect of compressor $\pi_c$

- As with turbojet, for very low compressor pressure ratio $\pi_c$, thermal efficiency is low, so both thrust and TSFC are low.
- At unrealistically high $\pi_c$, very little fuel can be added; after paying for compressor & fan work and expanding main stream, exit velocity $u_9 < u_1$ - negative thrust from main stream!

Turbofan - results - effect of $\pi_c$

- T-s diagrams same as turbojet except for tradeoff between KE of main stream and work going into fan.
- If low $\pi_c$ (or large $\alpha$ or $\pi_c'$), not enough turbine work to drive fan!

AME 436 - Spring 2016 - Lecture 13 - Ideal turbojets and turbofans
Turbofan - results - effect of $M_1$

- For high $M_1$, ST decreases since less fuel can be added ($\tau_\lambda$ limit)
- $\eta_{th}$ increases with $M_1$ since total pressure ratio $= \pi c\pi c'$ increases
- $\eta_{prop} = 2(u_1/u_9)/(1+ u_1/u_9)$ increases with $M_1$ since $u_1/u_9 \to 1$
- At very high $M_1$, high $\eta_{prop}$ anyway, no benefit of fan
- At very high $M_1$, again after paying for compressor & fan work and expanding main stream, $u_9 < u_1$ - negative thrust from main stream!

$M_1$ = varies
$\pi c = 30$
$\pi c' = 2$
$\tau_\lambda = 5$
$\alpha = 2$

Turbofan - results - effect of $\tau_\lambda$

- As $\tau_\lambda$ increases, ST increases but so does TSFC due to lower $\eta_{prop}$
- For high $\tau_\lambda$, a better approach would be to extract more turbine work to drive fan and thus get higher $\eta_{prop}$ - use higher $\alpha$ and/or $\pi c'$
- At very low $\tau_\lambda$, no heat addition is possible, thus no thrust

$M_1 = 0.8$
$\pi c = 30$
$\pi c' = 2$
$\tau_\lambda = \text{varies}$
$\alpha = 2$
Example - numerical

For an ideal turbofan with bypass ratio ($\alpha$) = 8, $\gamma = 1.35$, compressor pressure ratio ($\pi_c$) = 30, fan pressure ratio ($\pi_f$) = 1.8, flight Mach number ($M_f$) = 0.8, turbine inlet temperature = 1800K, ambient pressure ($P_a$) = 0.25 atm and ambient temperature ($T_a$) = 225 K, with $P_s = P_1$ and FAR << 1 determine:

(Note that most of these answers could be obtained from the formulas given earlier in this lecture without re-derivations, but this is needed for consistency with the example in the next lecture where non-ideal effects are considered and the simple formulas cannot be used.)

(a) $T$, $P$ and $M$ after the diffuser (station 2)

$M_2 = 0$ by assumption; $T_2 = T_{T_2} = T_2 + (1 + \frac{\gamma - 1}{2} M_2^2) \cdot (225) \cdot (1 + \frac{1.35 - 1}{2} 0.8^2) = 250.2K$

$p_2 = P_2 = P_0 \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right)^{-\frac{\gamma}{\gamma - 1}} = (0.25) \cdot (1 + \frac{1.35 - 1}{2} 0.8) \cdot 1.35^{0.35} = 0.377 \text{ atm} \$

(b) $T$, $P$ and $M$ after the compressor (station 3)

$M_3 = 0$ by assumption; $\pi_c = 30 = P_3/P_2 \Rightarrow P_3 = P_0 \cdot 30 \cdot 0.377 \text{ atm} = 11.3 \text{ atm}$

$t_3 = t_{T_3} = 250.2K \cdot 30 \cdot 0.35^{0.35} = 604.3 \text{ K}$

(c) $T$, $P$ and $M$ after the combustor (station 4)

$M_4 = 0$ by assumption; $P_{t_4} = P_4 = P_a = 11.3 \text{ atm}; T_{t_4} = T_4 = 1800K$ by assumption

Example - numerical (continued)

(d) $T$, $P$ and $M$ after the compressor turbine (station 5)

To balance turbine work with compressor work:

$C_p (T_{t_4} - T_{T_4}) = C_p (T_{T_4} - T_{t_4}) \Rightarrow 604.3K - 250.2K = 1800K - T_{t_4} = T_{t_4} = 1445.9K$

$\left( \frac{P_4}{P_3} \right)^{\frac{\gamma}{\gamma - 1}} \Rightarrow P_4 = P_3 \cdot 11.3 \text{ atm} = 1445.9K \cdot 0.35^{0.35} = 4.85 \text{ atm}; M_4 = 0$ by assumption

(c) $T$, $P$ and $M$ after the fan (station 6)

For the fan stream, $T_{t_2} = T_{T_2} = 250.2K$ and $P_{t_2} = P_2 = 0.377 \text{ atm}$.

Then we need to compute the fan stream work and equate it to the turbine work:

$W_{t,\text{fan}} = \dot{m}_f C_p (T_{t_2} - T_{T_2}) = -W_{t,\text{turbine}} = -\dot{m}_c C_p (T_{T_1} - T_{t_4}) \Rightarrow T_{T_1} = T_{t_4} = (\dot{m}_c / \dot{m}_f) \cdot (T_{T_2} - T_{T_1})$

$\Rightarrow T_{T_1} = T_{t_4} = (1445.9K + 8(250.2K - 291.4K)) = 1116.3K$

$\left( \frac{P_4}{P_{t_2}} \right)^{\frac{\gamma}{\gamma - 1}} \Rightarrow P_4 = 4.85 \text{ atm} \cdot \frac{1116.3K}{1445.9K} = 1.79 \text{ atm}$
Example - numerical (continued)

(c) T, P and M after the nozzle (station 9, and 9° for the fan stream) (note that for the fan stream, nothing happens between stations 3 and 6) 

\[
P_t = 0.25\text{atm by assumption}; \quad \frac{P_t}{P_s} = \left(1 + \frac{y-1}{2}M_t^2\right)^{\frac{y-1}{2}} \Rightarrow \frac{1.789\text{atm}}{0.25\text{atm}} = \left(1 + \frac{1.35 - 1}{2}M_t^2\right) \Rightarrow M_t = 1.95
\]

\[
T_{t,0} = T_{t,0}' = T_0\left(1 + \frac{y-1}{2}M_t^2\right) \Rightarrow T_{t,0} = T_{t,0}' = 1116.3K/\left(1 + \frac{1.35 - 1}{2}1.95^2\right) = 670.2K
\]

\[
P_t' = 0.25\text{atm by assumption}; \quad \frac{P_t'}{P_s'} = \left(1 + \frac{y-1}{2}M_t^2\right)^{\frac{y-1}{2}} \Rightarrow \frac{0.678\text{atm}}{0.25\text{atm}} = \left(1 + \frac{1.35 - 1}{2}M_t^2\right) \Rightarrow M_t' = 1.30
\]

\[
T_{t,0}' = T_{t,0}\left(1 + \frac{y-1}{2}M_t^2\right) \Rightarrow T_{t,0}' = T_{t,0}' = 291.4K\left(1 + \frac{1.35 - 1}{2}1.30^2\right) = 225K
\]

(f) Specific thrust (ST) (recall FAR << 1 and \(P_t = P_1\) by assumption)

\[
\text{Thrust} = m_v \left[u_v - u_t\right] + m_s \left[u_s - u_t\right]; \quad \text{ST} = \frac{\text{Thrust}}{\left(m_v + m_s\right) c_i} = \frac{m_v \left[u_v - u_t\right]}{\left(m_v + m_s\right) c_i} + \frac{m_s \left[u_s - u_t\right]}{\left(m_v + m_s\right) c_i}
\]

\[
\text{ST} = \frac{1}{1 + \alpha} \left[\left(M_s + \frac{T_s}{T_i} - M_s\right) + \alpha \left(M_s + \frac{T_s}{T_i} - M_s\right)\right]
\]

\[
\text{ST} = \frac{1}{1 + \alpha} \left[\frac{1}{1.95}\sqrt{\frac{670.2}{225} - 0.8} + 8\left(1.30\sqrt{225K}/225K - 0.8\right)\right] = 0.728
\]

Example - numerical (continued)

(g) TSFC and overall efficiency

Recalling that only 1/(1+\(\alpha\)) of the total air flow is burned,

\[
\text{TSFC} = \frac{m_v Q_v}{\text{Thrust} \cdot c_i} = \frac{m_v C_v (T_u - T_{t,0})}{\left(m_v + m_s\right) c_i} = \frac{m_v C_v (T_u - T_{t,0})}{\left(m_v + m_s\right) c_i} = \frac{\gamma}{ST \cdot 1 + \alpha \gamma T_i}
\]

\[
\text{TSFC} = \frac{1}{ST \cdot 1 + \alpha \gamma (\gamma - 1)T_i} \Rightarrow \frac{T_{t,0}}{T_{t,0}'} = \frac{1}{1 + 1800K - 604.3K} = 2.32; \quad \eta_S = \frac{M_t}{\text{TSFC}} = 0.8 \times 0.345
\]

(b) Propulsive efficiency

\[
\eta_p = \frac{\text{Thrust power}}{\Delta \text{(Kinetic energy)}} = \frac{\text{Thrust} \cdot u_t}{m_v \left(u_v^2 - u_t^2\right)/2 + m_s \left(u_s^2 - u_t^2\right)/2} = \frac{2 \left(m_v + m_s\right) c_i u_t}{\left(m_v + m_s\right) c_i \left(u_v^2 - u_t^2\right) + m_s \left(u_s^2 - u_t^2\right)}
\]

\[
\eta_p = \frac{\text{ST}}{M_s^2 c_i^2 - M_t^2 c_i^2 + \alpha \left(M_s^2 c_i^2 - M_t^2 c_i^2\right)} = \frac{\text{ST}}{M_s^2 (\gamma RT_{t,0}) - M_t^2 (\gamma RT_{t,0}) + \alpha (M_s^2 (\gamma RT_{t,0}) - M_t^2 (\gamma RT_{t,0}))}
\]

\[
\text{ST} = \frac{2(1 + \alpha) M_t T_i}{M_s^2 T_{t,0} - M_t^2 T_{t,0} + \alpha (M_t^2 T_{t,0} - M_t^2 T_{t,0})} = \frac{0.728}{1.95\sqrt{670.2K - 0.8^2225K} + 8(1.30^2225K - 0.8^2225K)} = 0.549
\]
(i) Thermal efficiency

\[ \eta_a = \frac{\Delta (\text{Kinetic energy})}{m\dot{Q}_b} = \frac{m\left(u_1^2 - u_2^2\right)/2 + m\left(u_3^2 - u_4^2\right)/2}{C_v(T_u - T_v)} = \frac{1}{2} \frac{C_v(T_u - T_v)}{C_v(T_u - T_v)} \]

\[ \eta_a = \frac{1}{2} \frac{\left(M_u^2 c_1^2 - M_0^2 c_1^2\right) + \alpha \left(M_u^2 c_2^2 - M_0^2 c_2^2\right)}{C_v(T_u - T_v)} = \frac{1}{2} \frac{C_v(T_u - T_v)}{C_v(T_u - T_v)} \]

\[ \eta_a = \frac{\gamma - 1}{2} \frac{\left(M_u^2 T_u - M_0^2 T_1\right) + \alpha \left(M_u^2 T_u - M_0^2 T_1\right)}{T_u - T_0} \]

\[ \eta_a = \frac{1.35 - 1 \left(1.95^5 670.2 K - 0.8^2 225 K\right) + 8 \left(1.30^5 225 K - 0.8^2 225 K\right)}{1800 K - 604.5 K} = 0.629 \]

Note that \( \eta_a \eta_b = (0.629)(0.549) = 0.345 = \eta_s \) as advertised

(j) Specific impulse

\[ I_s = \frac{1}{T \text{SFC}} \frac{\dot{Q}_b}{\dot{m}} = \frac{4.3 \times 10^7 J/\text{kg}}{0.52 \left(9.81 m/\text{sec}^2\right) \left(1.35 \left(8.314 J/moleK\right) \left(mole / 0.02897 \text{kg} \left(225 K\right)\right)\right)} = 6400 \text{sec} \]

Example - numerical (continued)

Example - graphical

In an ideal \( \tau_s \)-limited afterburning turbojet, how would the T-s diagrams be affected if

(a) A new turbine with higher \( \tau_s \) is used but \( \tau_s \text{AB} \) is unchanged.

The cycles are the same up to the end of compression process. The temperature after combustion increases. The compressor work and thus turbine work (thus \( \Delta T \)) does not change. Since \( \Delta T \) across the turbine does not change but the temperature at the start of the turbine expansion is higher than the base cycle, the pressure after expansion will be higher. Then more heat is added up to the afterburner limit.

(b) A new compressor is used with a higher pressure ratio

The compression process ends at a higher pressure and but the same entropy than the base cycle. Heat addition then occurs along this constant-pressure curve up to the turbine temperature limit, ending at lower entropy. More work, thus more \( \Delta T \), is required from the turbine, so the afterburning heat addition occurs along a lower constant pressure curve.
In an ideal turbofan, how would the T-s diagrams be affected if

(a) Ambient temperature is increased (ambient pressure not changed).

\[ T_1 \] increases and the starting point (1) moves up along the constant-pressure curve. Since the pressure ratios \( P_2/P_1 = (T_2/T_1)^{\gamma/\gamma} \) and \( P_3/P_2 = (T_3/T_2)^{\gamma/\gamma-1} \) are the same, \( T_2 \) and \( T_3 \) increase but \( P_2 \) and \( P_3 \) do not. The turbine inlet temperature limit, \( T_4 \) does not change so state 4’ is the same as 4. The increased compressor work must be paid for in the turbine. The fan work increases for the same reason as the compressor work assuming that the fan pressure ratio didn’t change.

(b) The flight Mach number is increased

\( T_2 \) increases and since \( P_3/P_2 = (T_3/T_2)^{\gamma/\gamma-1} \) is the same, \( T_3 \) increases. Then with the same turbine inlet temperature limit, \( T_4 \) does not change. The increased compressor work must be paid for in the turbine. The fan work increases for the same reason.

### Summary

- Turbojet
  - Ideal cycle - compress isentropically, burn isobarically (constant P), expand isentropically back to ambient pressure
  - Matching conditions: compressor work = turbine work & \( P_1 = P_9 \) (i.e. \( P_{exit} = P_{ambient} \))
  - Performance is determined by compressor pressure ratio \( \pi_c \), turbine inlet temperature limit (\( \tau_L \)) and flight Mach number (\( M_1 \))
    - Low \( \pi_c \): thermal efficiency low, thrust low
    - High \( \pi_c \): thermal efficiency high but not much fuel can be added before \( \tau_L \) limit reached so again thrust low (also propulsive efficiency low because of high exit M)
    - Intermediate \( \pi_c \): thermal efficiency intermediate, thrust highest
    - Increasing \( \tau_L \) always increases thrust but propulsive efficiency suffers

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Summary - continued

Afterburner
- Obtain more thrust by adding additional fuel after the turbine
- This is possible since $\tau_\lambda$ is too low for stoichiometric combustion, thus exhaust has residual $O_2$
- Additional parameter $\tau_{\lambda,AB}$
- Since there are no moving parts in the afterburner $\tau_{\lambda,AB} > \tau_\lambda$ and substantial additional fuel can be added
- T-s Carnot cycle strips are shorter for afterburning part of cycle, thus thermal efficiency is greatly reduced

Turbofan
- Used to increase propulsive efficiency by accelerating large mass of air by a smaller $\Delta u$
- Two streams - main (combusted) and fan
- Additional parameters $\alpha$, $\pi'_c$
- Fan work is "paid for" with additional turbine stages - lower $u_9$ for main stream
- Combined thrust maximized when most of thrust is obtained from fan stream