Theoretical Analysis of the Effect of Gravity on Premixed Turbulent Flames

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Abstract—The Bray-Moss-Libby model of premixed turbulent combustion supplemented with a model for the mean rate of creation of product is used to analyze the influence of gravity on horizontal premixed turbulent flames propagating either upward or downward into reactants. A parameter determining the extent of the influence of gravity involving various aerothermochemical quantities is identified. Earlier studies of gravity free flames are extended to include a prediction of turbulent flame speed including the influence of gravity by adoption of the KPP-condition of a minimum speed. A linearized treatment of the effect of gravity on the structure of premixed turbulent flames is carried out.

NOMENCLATURE

\begin{itemize}
  \item $A_1, B_1$: constants arising in the expansions about $\varepsilon = 0$ (cf. Eq. (21))
  \item $c$: progress variable
  \item $D_T$: Damköhler parameter (cf. Eqs. (10) and (11))
  \item $F$: dimensionless turbulent flux variable (cf. Eq. (12))
  \item $G, \bar{G}, \bar{G}^k$: gravity parameters (cf. Eqs. (16), (19) and (21))
  \item $h$: $h = h(\varepsilon)$ (cf. Eq. (15))
  \item $H$: gravity function (cf. Eq. (16))
  \item $I$: dimensionless velocity intensity (cf. Eq. (12))
  \item $I_0$: turbulent flame speed parameter
  \item $k_o$: turbulent kinetic energy in the reactant stream
  \item $K$: dimensionless third moment function (cf. Eqs. (13) and (14))
  \item $l_0$: integral length scale in the reactant stream
  \item $L$: dimensionless third moment function (cf. Eqs. (13) and (14))
  \item $P$: pressure
  \item $u$: $x$-wise velocity component
  \item $u_L$: laminar flame speed
  \item $\bar{u}_0$: turbulent flame speed
  \item $T$: temperature
  \item $\bar{T}$: integral time scale of flamelet crossings
  \item $\dot{\gamma}$: mean rate of creation of product
  \item $x$: coordinate normal to the flame
  \item $\alpha$: exponent in the expansion about $\varepsilon = 1$
  \item $\beta_0, \beta_1, \gamma_0, \gamma_1$: coefficients in the expansions about $\varepsilon = 0, 1$
  \item $\chi_L, \chi_{\nu}$: dissipation terms
  \item $\lambda$: empirical constant $\lambda = \kappa_1 - \phi_n$
  \item $\nu$: flamelet crossing frequency
  \item $\phi_n$: parameter arising from the velocity-chemical creation correlation (cf. Eq. (9))
\end{itemize}
INTRODUCTION

It can be expected that horizontal premixed turbulent flames propagating upward into dense reactants can under some circumstances behave considerably differently from such flames propagating downward into those same reactants. In one case the gravitational vector is codirectional with the mean density gradient while in the second case it is oppositely directed. Recent work (Bray et al., 1981; Masuya and Libby, 1981; and Libby, 1985) showing that mean force fields interact with density fluctuations to influence significantly the transport processes and the turbulence production in premixed turbulent flames suggests that the body force due to gravity should likewise have a similar influence under appropriate circumstances. The purpose of the present study is to analyze the effect of gravity on premixed turbulent flames, their rate of propagation and structure and thereby to determine the circumstances under which that effect is significant.

The literature concerning acceleration effects in general and gravitational effects in particular on such flames appears to be exceedingly sparse. Bray and Moss (1981) use a classical gradient transport analysis to show that a premixed turbulent flame can propagate at a considerably higher than normal velocity if subjected to an acceleration which is codirectional with the mean density gradient. Motivation for their study is provided by observations of extraordinary propagation speeds for turbulent flames associated with industrial accidents. An extensive study of such flames is provided by Gugan (1978).

Strehlow and coworkers (see Strehlow, 1983) report the results of experiments involving the unconfined propagation of flames in premixed gases. A hemispherical bag made of a thin film is anchored on the ground and filled with reactants which at the beginning of the test are centrally ignited on the ground plane. A roughly hemispherical flame propagates into the reactants and soon tears the confining bag from its constraints. Motion pictures permit the mean flame trajectory and flame character to be analyzed. In a series of tests the flames are found to be weakly turbulent and to exhibit gravitational effects. The horizontal portion of the flame on the top of the hemisphere propagates into the reactants at from 1.3 to 1.7 times the speed of the vertical portion of the flame along the ground plane. Thus these data are indicative of gravitational effects but not adequate for quantitative analysis.

Lewis (1973) describes the results of an experiment involving premixed turbulent flames advancing radially inward in a tube which is rotating in a horizontal plane about a vertical axis through its mid-point. By varying the rotational speed of the tube the flames are subjected to various degrees of acceleration codirectional with their mean density gradient. No turbulence is generated prior to ignition so the turbulence is generated by the flow induced by the expansion of the product gases. Results are presented for several hydrogen-air and propane-air mixtures in terms of observed
flame speed relative to the tube versus the centrifugal force. No significant influence of acceleration is found until the centrifugal force is roughly $10^2 \text{g}$. Increases beyond this value result in apparent quadratic increases in flame speed until extinction occurs but there is sufficient scatter in the data that a linear increase in flame speed with acceleration can be rationalized.*

Several comments concerning the results of Lewis (1973) are indicated. The observed flame speed is not the usual turbulent flame speed because the reactants are compressed and moved radially inward as a consequence of the lower density of the products. Thus the former speed is a combination of inward reactant flow and the turbulent flame speed. In addition turbulent flame speeds are usually expressed in terms of the ratio $\tilde{u}_0/k_\theta^{1/2}$ or a related quotient where $\tilde{u}_0$ is the flame speed and $k_\theta$ is the turbulent kinetic energy in the reactant stream approaching the flame.† In the experiments of Lewis this energy is uncertain; it is not zero because of the compression of reactants noted earlier results in boundary layers on the walls of the tube and because upstream, pressure-induced fluctuations arise in the reactant stream from the turbulence in the products. It is thus reasonable to suggest that the insensitivity of the observed flame speed for low to moderate centrifugal forces may be associated with nearly constant values of the quotient $\tilde{u}_0/k_\theta^{1/2}$ while at least part of the observed increase of that speed for high forces is due to increases in $k_\theta$. The implication from these comments is that the data from Lewis (1973) regarding the influence of acceleration on turbulent flame speed are quantitatively ambiguous but are qualitatively in accord with expectations.

The central theoretical problem involved in assessing the influence of gravity on premixed turbulent flames can be focused on the following question: Given a flame undergoing a constant gravitational acceleration and propagating into reactants with a specified turbulent kinetic energy and turbulent length scale determine the speed and structure of the flame. It is this problem which we consider here by applying the Bray–Moss–Libby (BML) model of premixed combustion (cf. Bray et al., 1981 and Libby, 1985) to horizontal planar flames subject to gravity.

It benefits our subsequent exposition to set forth in a general way the regime of premixed turbulent combustion amenable to treatment by the BML-theory. A widely accepted means for delimiting those regimes is the Borghi diagram (cf., e.g., Peters (1986)) which considers the quantities identifying the turbulence and combustion: the characteristic velocities, $u'_c$ and $u'_\tau$, the rms intensity in the reactant stream and the laminar flame speed respectively, and the characteristic lengths, $l_0$ and $l_L$, the turbulence integral scale and the laminar flame thickness respectively. These quantities can be used to define several dimensionless parameters: a turbulence Reynolds number $Re \equiv u'_c l_0/\nu$, a turbulence Damköhler number $Da \equiv u'_\tau l_0/\nu u'_c l_L$ and a turbulence Karlovitz number $Ka \equiv Re^{1/2}/Da$. In the Borghi diagram the quotients $u'_c/\nu$ and $l_0/l_L$ are related by lines of constant $Re$, $Da$ and $Ka$, lines which are conveniently straight in a log-log plane. The flamelet regime in which the BML theory applies is identified by $Re \gg 1$ and $Da \gg 1$ subject to the restriction that $Ka < 1$. This restriction implies that the flame thickness must be smaller than the Kolmogoroff length, i.e., that the Klimov–Williams criterion applies, and that the stretch to which the flamelets are subjected must be suitably weak. The implication is that loosely the BML-theory applies to flows involving vigorous turbulence and fast chemistry but with a restraint such that the flamelets are not excessively stretched.

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*The author is indebted to K.N.C. Bray for pointing out this possibility.
†But see our later discussion regarding the variability of $\tilde{u}_0/k_\theta^{1/2}$.
The BML-model has been successfully applied to infinite normal (Bray et al., 1981) and oblique (cf. Masuya and Libby, 1981) flames and has exposed conceptually important phenomena such as countergradient and nongradient diffusion and turbulence production due to the interaction between density fluctuations and mean force fields arising from gradients of mean pressure and Reynolds shear stress. A shortcoming of these applications is the inability of the analysis to predict turbulent flame speeds with the consequence that a parameter, denoted $\tilde{u}_0$ and related to the quotient $\tilde{u}_0/k_0$ discussed earlier, is assumed given. Clearly this shortcoming must be remedied before one objective of the present study, viz to predict the influence of gravity on turbulent flame speed, can be achieved.

The inability to predict flame speed is a shortcoming of many flame theories. The "cold boundary difficulty" (cf., e.g., Williams 1985, 145-149) in laminar flame theory is a well known example of the need to bring supplementary information to bear before a propagation speed is predicted. However, there is an extensive literature related to these shortcomings and to remedies therefor. Of relevance to the present analysis is the Kolomogorov-Petrovsky-Piskunov (KPP) theory (cf., eg., Clavin and Linan 1984, Section II.2.2) which can be interpreted as follows: Of the theoretically possible either continuous or discrete spectrum of flame speeds only the lowest is relevant for the steady propagation of a flame. The KPP-theory shows that for a class of reaction rate functions the speed of a flame is determined by the flow at the cold, i.e., leading, edge of the flame. Given the clear interdependence of the reactant and product streams in turbulent flames as suggested by both physical and mathematical considerations, that $\tilde{u}_0$ should be determined by conditions at one edge of the flame may be disquieting. However, in a study of laminar flame propagation in a medium that reacts at its initial temperature Zeldovitch (1980) shows the importance of the leading edge of the flame in determining flame speed. Moreover, we might expect the cold boundary to serve a flame holding function in turbulence. Finally, since the distribution of the mean rate of creation of product across a turbulent flame is likely to correspond to that for which the KPP-theory applies, i.e., a relatively broad distribution of the mean rate of creation with mean temperature, we here examine for the first time the implications of the theory relative to a BML-description of horizontal premixed turbulent flames. To justify this examination we promptly summarize our findings: A prediction of the speed of turbulent flames free of gravitational effects in accord with some but not all of the available experimental data is obtained. In addition a plausible prediction of the effect of gravity on flame speed including extinction under suitable conditions is given. Thus the present application of the KPP-theory leads to useful results.

Two related works deserve mention. Hakberg and Gosman (1984) apply the KPP-theory to a classical gradient transport description of premixed turbulent flames similar to that of Bray and Libby (1976). They show that $\tilde{u}_0/k_0^{1/2}$ is constant for a wide range of conditions with a value again in agreement with some but not all of the available data. Professor K. N. C. Bray has informed the author in a private communication of finding in unpublished work involving characteristic solutions for the propagation of one-dimensional premixed turbulent flames whose aerothermochemistry is described by the BML-model that flame speed is determined by conditions at the cold boundary.

Our presentation is organized as follows: Employing the BML-model for the aerothermochemistry, we analyze in the next section a horizontal premixed turbulent flame including the influence of gravity. A recently developed model for the mean rate of creation of product is introduced and leads to the identification of a parameter characterizing the influence of gravity. The describing equations lead naturally to a
parameter identifying the circumstances under which gravitational effects are significant. Application of the KPP-theory is shown to determine \( l_0 \) and to lead to plausible results. To simplify the numerical analysis we solve the resulting equations for small values of the gravity parameter and present numerical results for ranges of the parameters of applied interest. We close with a discussion of the implications of our findings.

2 ANALYSIS

Figure 1 shows the significant variables associated with a horizontal premixed turbulent propagating upward into cold reactants and depositing behind hot products, i.e., with an acceleration vector codirectional with the \( x \)-coordinate. If we change the sign of \( g \) we have the reverse situation, a horizontal flame propagating downward into reactants.

According to the Bray–Moss (BM) model of the thermochemistry the entire state of the gas is determined by a progress variable \( c(x, t) \) with the value zero in reactants, unity in products and intermediate values in gas possibly undergoing chemical reaction. In this case the equation of state is approximated by

\[
\frac{\rho}{\rho_r} = \frac{1}{1 + \tau c} \frac{T}{T_r} = 1 + \tau c
\]

where the subscript \( r \) denotes reactants and \( \tau \) is a heat release parameter whose value is readily determined from the temperature in the products, i.e., form

\[
\frac{T}{T_r} = 1 + \tau
\]

Values of \( \tau \) of applied interest are generally about six but can range from three to nine.

Central to the BM-model and its derivative, the BML-model, is the assumption that reaction takes place in thin surfaces such that states corresponding to values of the

![FIGURE 1 Schematic representation of a horizontal premixed turbulent flame.](image-url)
progress variable intermediate between zero and unity are rare and thus that the probability density of that variable is essentially bimodal. The consequence of this assumption is that the modeling necessary to achieve closure of a second moment formulation is greatly simplified.

2.1 The Conservation Equations

For the flame shown in Figure 1 the describing equations in terms of Favre averages are as follows (cf., e.g., Libby (1985))

\[
\frac{d}{dx} \bar{\rho} \bar{u} = 0
\]

\[
\frac{d}{dx} \left( \bar{\rho} \bar{u}^2 + \bar{u} \bar{c} \right) = - \frac{dp}{dx} + \bar{g}
\]

\[
\frac{d}{dx} \left( \bar{\rho} \bar{c} \bar{u} + \bar{u} \bar{c} \bar{u} \bar{c} \right) = \bar{w}
\]

\[
\frac{d}{dx} \left( \bar{\rho} \bar{u} \bar{c} \bar{c} + \bar{u} \bar{c} \bar{c} \right) + 2 \bar{u} \bar{c} \bar{c} \frac{du}{dx} + 2 \bar{u} \bar{c} \bar{c} \frac{dc}{dx} = -2 \bar{u} \frac{dp}{dx} - \chi_w
\]

Several comments regarding these equations are indicated. The effect of gravity is contained in the one term on the right side of Eq. (4). If conventional rather than Favre averaging is employed, there would appear on the right side of the \( \bar{u} \bar{c} \bar{c} \frac{dp}{dx} \) term in place of the \( \bar{u} \bar{c} \bar{c} \frac{dp}{dx} \). Thus gravitational effects appear differently in the two formulations. In these equations the influence of pressure fluctuations is neglected compared with that of the mean pressure gradient; given the thinness of the flame typically on the order of the integral length of the turbulence in the reactant stream, this influence is believed to be small and in any case no satisfactory model therefore exists. These equations apply to flows involving continuous variations of density but in the present application the two widely different densities, those of reactants and products, are separated by thin interfaces, the laminar flamelets, which advance into the reactants at the laminar flame speed \( \bar{u}_f \). Thus the usual mechanisms of interaction of gravity and density inhomogeneities would appear not to apply. Several of the terms obviously calling for modeling in these equations, \( \bar{c} \bar{w} \), \( \bar{c}^2 \) and \( \bar{u}\bar{w} \), are readily obtained from the BM-model. In particular we have

\[
\bar{u}^* = \bar{u} \frac{\bar{u} \bar{c} \bar{c}}{\bar{Q}}
\]

\[
\bar{c}^* = \bar{c} \frac{\bar{u} \bar{c}^2}{\bar{Q}} = \bar{c} (1 - \bar{c}) \frac{1}{1 + \tau \bar{c}}
\]

\[
\bar{u}^* \bar{w} = -\bar{w} \frac{\bar{u} \bar{c} \bar{c}}{\bar{Q} (1 - \bar{c})} (\bar{c} - \phi_n)
\]
where \( \phi_n \) is a parameter dependent on the description of the instantaneous rate of creation of product \( \bar{w}(c) \), taken as in our earlier studies to have the value 0.833. The modeling of the third moment quantities in Eqs. (6) and (7) is greatly facilitated as will be shown later by the Bray–Moss–Libby (BML) model of the aerothermochemistry of premixed turbulent combustion. We are thus left to consider the dissipation terms \( \chi_u \) and \( \chi_w \), and the mean rate of creation of product \( \bar{w} \). The modeling of these terms is also discussed later.

Under the assumption that the flame is thin we impose conditions on the solutions to these equations at \( \pm \infty \), i.e., we assume that the flame is between two regions of essentially uniform mean velocity and uniform turbulence, each subject to a mean pressure gradient resulting in hydrostatic equilibrium with densities \( \rho_1 \) and \( \rho_2 \).

### 2.2 The Mean Rate of Creation of Product

In previous studies of normal premixed turbulent flames by means of the BML-model the calculations are carried out with the mean progress variable \( \bar{c}(x) \) as the independent variable and indeed comparison between theory and experiment is conveniently performed in terms of this variable. In this case no model is needed for the mean rate of creation of product, i.e., for \( \bar{w}(x) \) and a significant difficulty is avoided. Since the presence of the gravitational term in Eq. (4) results in a requirement for a model for \( \bar{w}(x) \) in the present analysis, it is fortunate that an appropriate one has recently become available. An analysis of the statistical geometry of the idealized progress variable \( \bar{c}(t; x) \) which can be obtained experimentally from a temperature signal \( T(t; x) \) suitably interpreted as a two-valued function, \( T_1(t; x) \) and \( T_2(t; x) \), is given by Bray et al. (1984) and Bray and Libby (1986). The connection between this analysis and \( \bar{w}(x) \) resides in the simple relation

\[
\bar{w}(x) = w_r(x) v(x)
\]

where \( w_r \) is the mean rate of creation of product per flamelet crossing in units of density and \( v \) is the frequency of flamelet crossings, i.e., the number of 0–1 and 1–0 changes per unit time. The first factor on the right side depends on the distribution of the rate of strain experienced by the flamelets at various locations within the turbulent flame (cf. Bray 1986). For weakly and moderately strained flamelets \( w_r \) may be treated as constant since as noted earlier the BML-theory applies when the Karlovitz number is less than unity, i.e., when the flamelets are subject to only modest rates of stretch.

The second factor in Eq. (9) is given in a recent paper by Bray et al. (1988) as

\[
v(x) = \frac{(1 + \tau)\bar{c}(1 - \bar{c})}{\bar{T}(1 + \tau\bar{c})^2}
\]

where \( \bar{T}(x) \) is an integral time scale determined by the time autocorrelation of the two valued progress variable, a quantity which at present must be obtained from experiment. In the cited reference the presently available data relative to \( \bar{T} \) are reviewed and it is concluded that for the normal flames considered here \( \bar{T} \) can be reasonably assumed to be a constant across the flame. We shall find later that the model for \( \bar{w}(x) \) based on Eqs. (8) and (9) leads in the present application to a Damköhler parameter defined as

\[
D_T = \frac{w_r\bar{T}_0}{\rho_1\bar{T}_u \bar{w}_0}
\]
where \( l_0 \) is the integral scale of the turbulence in the reactant stream approaching the flame and \( u'_0 \equiv (u_{0}^{2})^{1/2} \) is the r.m.s. of the intensity in that stream, quantities introduced earlier.

It will be convenient in the identification of the parameter characterizing the influence of gravity to have an estimate for both \( w_f \) and \( T \). Although there is at present no adequate theory for either of these quantities, reasonable estimates for the normal flames under consideration are given in Bray et al. (1987) as follows:

\[
\frac{w_f}{q_r} = \frac{u_t}{u_0} \quad T = \frac{l_0}{u_0}
\]

where \( u_t \) is the classical laminar flame speed in the chemical system under consideration. If these estimates are substituted into Eq. (10), we have

\[
D_r = \frac{u_t}{u_0}
\]

an equation we use to eliminate the parameter \( D_r \).

2.3 The Describing Equations

The principle dependent variables arising in the analysis of Eqs. (3)-(7) are

\[
I \equiv \frac{q u'^2}{q_r u'_0} \quad F \equiv \frac{q u' c'}{q_r u'_0}
\]

so that \( I \) is a dimensionless intensity with the value \( l_0 \) in the reactant stream approaching the flame as mentioned earlier and \( F \) is a dimensionless flux of the progress variable and according to the BM-model directly related to the flux of any state variable.

Two other auxiliary variables arise, namely

\[
L = \frac{q u'^3}{q_r u'_0} \quad K = \frac{q u'^2 c'}{q_r u'_0}
\]

These are dimensionless forms of the third moment quantities in the second moment equations. With approximations appropriate for the present study these are given by the BML-model as

\[
L = (1 - 2\bar{\varepsilon}) h^2 F^3
\]

\[
K = (1 - 2\bar{\varepsilon}) h F^2
\]

Equations (14) incorporate two approximations involving the conditional statistics of the velocity components within reactants and products. The first is that the conditional probability density functions (pdf's) are Gaussian so that the conditional third moments are zero. Recent experimental data from Driscoll and Gulati (1987) and computational results from Anand and Pope (1986) show that this assumption is not completely correct and that the pdf's in question are skewed toward low
velocities but with unimportant consequences. A second approximation in Eqs. (14) relates to neglect of the differences in the conditional intensities. Various models for these intensities have been employed in earlier work (cf. Libby, 1985) but none appear to be generally valid and in any case the experimental data concerning them is highly uncertain. Fortunately, the differences in question do not appear to play an important role in these third moment quantities. The implication flowing from these two approximations is that the unconditional third moments are dominated by fluctuations between the conditional mean values within reactants and products, a dominance considered to be valid for present purposes.

As in earlier studies the dissipation terms are taken to describe processes associated with chemical reaction and not with conventional molecular effects which require distances in the streamwise direction considerably greater than the thickness of the flames under consideration in order to become operative. Thus

\[ x_u = \kappa_1 \bar{\omega} \left( \frac{\bar{\omega} c'}{\bar{c}(1 - \bar{c})} \right)^2 \]

\[ x_{uc} = \kappa_2 \bar{\omega} \left( \frac{\bar{\omega} c'}{\bar{c}(1 - \bar{c})} \right) \]

where \( \kappa_1 \) and \( \kappa_2 \) are empirical constants discussed later.

With these preliminaries we are able to write the principal equations to be dealt with, namely

\[ \left[ (1 + \tau \bar{c}) I' + L' + 2 \tau I \right] = 2 \tau F (I' + 1 - H) - \kappa_3 (1 + F') (hF)^2 \]

\[ \left[ (1 + \tau \bar{c}) F' + K' + I \right] = \frac{\tau}{h} (I' + 1 - H) - (1 + F') h F (\bar{c} + \lambda) \]

where prime now denotes differentiation with respect to \( \bar{c} \), taken to be the independent variable. Here \( \lambda = \kappa_1 - \phi_0 \). In earlier studies the value of \( \lambda \) is taken to be 0.017 while that of \( \kappa_2 \) is either zero or 0.25. The quantity \( h \) is introduced for notational convenience and is defined as

\[ h = \frac{1 + \tau \bar{c}}{\bar{c}(1 - \bar{c})} \]

Note that \( hF \) is finite as \( \bar{c} \to \pm 0, 1 \) so that the singularities in \( h \) are benign.

These equations differ from those treated earlier only in the \( H \)-functions in the first terms on the right side of each equation, a function defined as

\[ H = \frac{G F^{1/2}}{1 + \tau} (1 + F') \]

where

\[ G = \frac{gl_0}{\bar{u}_0 u_0} = \frac{gl_0}{u_0 \bar{u}_0} \]
is a gravity parameter. The second, more convenient form for \( G \) arises from use of Eq. (11) to eliminate \( D_T \). We see that the influence of gravity as reflected in \( G \) depends on the large turbulence scales via \( \lambda_0 \) and on two velocities, namely the laminar flame speed and the r.m.s. fluctuations of the velocity in the reactant stream. Such an influence can thus be expected to be significant for flames with large turbulence scales and small values of the product \( u_1 u_\alpha \), i.e., for lean and rich gas mixtures in low intensity turbulence, plausible dependencies.

If the structure of the flame in terms of the space variable \( x \) is desired, we must use Eq. (5) to compute \( d\bar{c}/dx \) and must integrate with respect to \( \bar{c} \) to obtain in dimensionless form

\[
\frac{x}{l_0} = \frac{u'_0}{\bar{u}_1 l_0^{1/2}} \int_{0}^{\bar{c}} \frac{(1 + r\bar{c}) (1 + F') \, d\bar{c}}{(1 + r) \bar{c} (1 - \bar{c})} (17)
\]

where without loss of generality we place the origin at \( \bar{c} = 1/2 \). The appearance of the factor \( \bar{c} (1 - \bar{c}) \) in the denominator of the integrand assures that the integral is divergent as \( \bar{c} \to 0, 1 \) as required in order for \( x \to \pm \infty \).

2.4 The Boundary Conditions

Equations (15) are to be solved subject to boundary conditions at \( \bar{c} = 0, 1 \). The repeated appearance of the factor \( \bar{c} (1 - \bar{c}) \) in the denominator in various terms implies that series expansions about the ends of the range of integration is called for in order to initiate the integration at points interior to that range. For \( \bar{c} \to 0 \) we let

\[
I \approx I_0 + \beta_0 \bar{c} + \ldots
\]

\[
F \approx \gamma_0 \bar{c} + \ldots
\]

where \( \beta_0 \) and \( \gamma_0 \) are determined by substitution into Eqs. (15) and by collection of the constant terms. We find

\[
\gamma_0 = \frac{1}{2} \left[ - (1 + \hat{G}) + (1 + \hat{G})^2 \right] - 4 \left( \frac{I_0}{1 + \lambda} + \hat{G} \right)^{1/2}
\]

\[
\beta_0 = -(1 + \kappa_2) \gamma_0 - \kappa_2 \gamma_0^2 - 3\pi I_0 - 2\gamma_0 (1 + \gamma_0) (1 + \lambda) \hat{G}
\]

where

\[
\hat{G} = \frac{\tau G l_0^{1/2}}{(1 + r)(1 + \lambda)}
\]

As \( \bar{c} \to 1 \) a more elaborate series is required. In this case we let

\[
I \approx I_\alpha + \beta_1 (1 - \bar{c}) + A_1 B_1 (1 - \bar{c})^{n_1} + \ldots
\]

\[
F \approx \gamma_1 (1 - \bar{c}) + B_1 (1 - \bar{c})^{n_1}
\]

where \( I_\alpha \) and \( B_1 \) are arbitrary but \( \beta_1, A_1, \alpha_1 \) and \( \gamma_1 \) are determined by substitution into Eq. (15) and collection of the constant terms and the \( (1 - \bar{c})^{n_1-1} \) terms.
We find

\[
\gamma_i = \frac{1}{2} \left[ - (1 + \tilde{G}) + \left( 1 + \tilde{G} \right)^i + 4 \left( \frac{I_0}{(1 + \tau)\lambda} + \tilde{G} \right)^{1/2} \right]
\]

\[
\beta_i = (1 - \kappa_i)(1 + \tau)\gamma_i^3 + \kappa_i(1 + \tau)\gamma_i^2 + \frac{3\pi I_0}{1 + \tau} + 2\lambda \gamma_i (1 - \gamma_i) \tilde{G}
\]

\[
\alpha_i = \frac{1 + \lambda}{1 + (\lambda - 1)\gamma_i - \lambda \tilde{G}}
\]

\[
A_i = \frac{1}{\alpha_i} \left[ (2\kappa_2 \gamma_i - 2\kappa_1 \gamma_i^3)(1 + \tau) + 2\lambda (1 - \gamma_i(1 + \alpha_i)) \tilde{G} \right] - (\kappa_2 - 3)(1 + \tau)\gamma_i^2
\]

where

\[
\tilde{G} = \frac{\tau GI_0^{1/2}}{(1 + \tau)}
\]

The solutions proportional to \((1 - \tilde{c})^n\) are eigenfunctions which are subject to benign restrictions discussed in detail in Libby (1985). In brief for the present series to apply \(z_i\) must be within the range one to two.

Equations (18)-(21) are interpreted as follows: If \(I_0\) is known, an integration of Eqs. (15) in the direction of increasing \(\tilde{c}\) can be initiated from a small positive value of \(\tilde{c}\). Similarly, if \(I_0\) and \(B_i\) are known, a second integration can be started for \(\tilde{c}\) less than, but close to, \(\tilde{c} = 1\). This interpretation provides three quantities \(I_0\), \(I_0\) and \(B_i\) to make the two independent variables \(I\) and \(F\) continuous at the terminal point of the two integrations. Clearly, we have an undetermined problem; it is this circumstance which leads in our earlier studies to the specification of \(I_0\) by reference to experimental data. For the purposes of those studies, namely the exposure of countergradient diffusion, nongradient diffusion and turbulence production in premixed turbulent flames, this strategy is appropriate since it assures that the predictions relate to experimentally observable flames but for the present investigation an alternative strategy is required.

2.5 The Values of \(I_0\)

At this juncture it is appropriate to discuss the data relative to \(I_0\). In our earlier studies we have taken either \(I_0 = 0.22\) as suggested by a variety of data (cf. e.g., Libby 1985) or \(I_0 = 0.16\) obtained in an open flame by Moss (1980). However, a wide range of values of \(I_0\) has recently appeared in the literature. For example, Cheng and Shepherd (1986) find from their experiments on v-flames stabilized on a rod various values down to \(I_0 = 0.02\). In a similar but somewhat different experiment Driscoll and Gulati (1987) obtain values as low as \(I_0 = 0.06\). Anand and Pope (1987) calculate by a pdf method \(I_0 = 0.44\) for values of \(\tau\) of applied interest while Hakberg and Gosman (1984) calculate a range of values depending on the details of their analysis, namely \(0.35 < I_0 < 0.65\), and quote further experimental results within this range.* Thus a

*Gunther (1983) provides a broad review of experimental techniques and data relative to turbulent flames and inter alia discusses the significant scatter for \(I_0\).
theorician expecting to compare an *a priori* prediction of flame speed with experimental data and with earlier predictions confronts significant uncertainty. Whether this range is due to differences in flame configuration leading to deviations from one dimensionality, e.g., those obviously prevailing in the fan flames of Cheng and Shepher and Driscoll and Gulati, or to other aerothermochernical differences in the various experiments is unknown. Whether the range of *predicted* values is due to differences in the models used is likewise unknown. For present purposes we can only assume that \( I_0 = 0.22 \) is a representative value for planar premixed turbulent flames free of gravitational effects.

2.6 *Application of the KPP-Theory and the Consequences*

We now examine the implications of the KPP-theory to the prediction of flame spread in the context of the present analysis. As noted earlier the essential conclusion of the theory is that of the spectrum of predicted flame speeds the minimum is the physically relevant one for steady state flame propagation. A restriction on the maximum value of \( I_0 \) is clearly imposed by the first of Eqs. (19), namely that the radicand must be nonnegative. Since a minimum flame speed corresponds to a maximum in \( I_0 \), denoted \( I_{0_{cr}} \), the KPP-theory readily yields

\[
I_{0_{cr}} = \left[ \frac{\tau G}{(1 + \tau)(1 + \lambda)} + \frac{2}{(1 + \lambda)^{1/2}} \right]^{-2}
\]

where we have selected the plus sign to obtain a single function for \( I_{0_{cr}} \) over the entire range of permissible values of \( G \).

Equation (22) clearly calls for comment. If \( G = 0 \), this equation gives

\[
I_{0_{cr}} = \frac{1 + \lambda}{4}
\]

Since our earlier studies we find that \( \lambda \ll 1 \) yields satisfactory agreement with the available data on the structure of premixed turbulent flames, this result is remarkably close to the value we have accepted as representative for gravity-free flames. If

\[
\frac{\tau G}{1 + \tau} = -2(1 + \lambda)^{1/2}
\]

then \( I_{0_{cr}} \to \infty \) implying that \( \hat{u}_0 \to 0 \) and that extinction occurs for a premixed turbulent flame propagating downward into reactants if \( I_0/\hat{u}_L \hat{u}_0^{*} \) is suitably large. That our theory does not predict degeneration to a laminar flame speed rather than \( \hat{u}_0 = 0 \) is a consequence of our neglect of molecular transport in Eqs. (4)-(7). Finally, as \( G \to \infty, I_{0_{cr}} \to 0, \hat{u}_0/\hat{u}_L \to \infty \), i.e., the turbulent flame speed increases indefinitely with increases in \( I_0/\hat{u}_L \hat{u}_0^{*} \). Since increases in flame speed are expected to result in accompanying increases in the intensity of the velocity fluctuations within the flame and in the rates of strain experienced by the laminar flamelets, in due course the treatment of \( w_p \) as constant will be inappropriate with the consequence that some self-limiting flame speed will develop.

These are all plausible results which unfortunately cannot be compared quantitatively with experimental data, given the absence of such data, but which nevertheless encourage examination of the effect of gravity on the structure of premixed turbulent...
flames with speeds determined by Eq. (22).* Despite these encouraging results we must make the following observation: When converted to a relation for the turbulent flame speed in the usual form of $\tilde{u}_0/\tilde{u}_0'$, Eq. (22) yields a linear dependence with the gravity parameter $G$, namely

$$\frac{\tilde{u}_0}{\tilde{u}_0'} = \frac{2}{(1 + \lambda)^{1/2}} + \frac{\tau G}{(1 + \tau)(1 + \lambda)}$$

(23)

Such a variant is in agreement with neither the theoretical predictions of Bray and Moss (1981) based on a classical gradient transport calculation nor the quadratic effect of acceleration inferred by Lewis (1973) on the basis of highly scattered data. Moreover, we pointed out earlier that because the propagation of a flame in a close tube is involved, increases in $\tilde{u}_0$ lead to corresponding increases in $\tilde{u}_0'$ so that a reinterpretation of these data tends to diminish the sensitivity of flame speed to acceleration.

The determination of $I_0$ by Eq. (22) significantly influences details of the solutions to Eqs. (15) without corresponding changes in the predictions relative to gravity-free flames. The first of Eqs. (19) implies that

$$\gamma_0 = -\frac{1}{2}(1 + \tilde{G})$$

(24)

and as $G \to \infty$, then $\gamma_0 \to -1$, the limiting value of $\gamma_0$. With $G < 0$ another possible restriction on the minimum in $G$ arises. If

$$\frac{\tau G}{1 + \tau} = -(1 + \lambda)^{1/2}$$

(25)

i.e., at a value of $G$ prior to extinction, then $\gamma_0 = 0$. For smaller values of $G$ countergradient diffusion is predicted to occur at the cold edge of the flame with the consequence that the flame holding capabilities of the cold edge may be considered to be lost but whether this additional requirement is significant is unknown. Certainly the notion that at suitably large values of $-\tau G/(1 + \tau)$ extinction occurs for premixed turbulent flames is appealing but the value at which extinction occurs is uncertain. Furthermore, if $I_0$ is determined either from experimental data or from Eq. (22), the present modeling leads to a simplified strategy of numerical solution in that the integration from the neighborhood of $\tilde{e} = 0$ can be carried out free of unknown parameters provided $\lambda$ and $\kappa_2$ are specified. Iteration is then confined to the determination of $I_0$ and $B_1$ so that an integration from the neighborhood of $\tilde{e} = 1$ leads to continuous solutions at the terminal point of the two integrations.

2.6 Linearization of the Effect of Gravity

Equations (15) subject to the boundary conditions given by Eqs. (18)-(21) with $I_0$ given by Eq. (22) present a difficult two point boundary value problem involving constraints within the 0–1 range of integration. In particular $I(\tilde{e})$ and the conditional intensities must be nonnegative for all $\tilde{e}$ with only $\lambda$ and $\kappa_2$ available to satisfy this constraint. Although we have not examined in detail the singular point of Eqs. (15),

*It is worth noting that in the course of this study an alternative to the KPP-theory, namely to impose explicitly a gradient transport model at the cold edge of the flame, was examined but found to impose a restriction on an empirical constant in that model rather than to determine $I_0$. 
our experience with numerical results suggests that for $G = 0$ there exists a relatively small domain in $\lambda - \kappa_2$-space yielding solutions satisfying the boundary conditions at both ends of the integration interval and that with $G \neq 0$ this domain changes. On physical grounds it would be highly desirable if these parameters were fixed for all values of $\tau$ and $G$ rather than adjusted from case to case. Thus we confront significant numerical and conceptual difficulties. To simplify the numerical analysis we adopt an alternative strategy adequate for present purposes, namely we deal only with suitably small values of the gravity parameter $G$ so that it may be considered an expansion parameter and select $\lambda$ and $\kappa_2$ so that satisfactory results for gravity-free flames are obtained. A thorough study of Eqs. (15) for arbitrary values of $G$ is not indicated until appropriate experimental data on the influence of gravity on premixed turbulent flames are available.

To this end consider the following series expansions:

$$I(\tilde{\epsilon} ; G) \approx (0)I(\tilde{\epsilon}) + (1)I(\tilde{\epsilon})G + \ldots \quad (26)$$

$$F(\tilde{\epsilon} ; G) \approx (0)F(\tilde{\epsilon}) + (1)F(\tilde{\epsilon})G + \ldots$$

and the auxiliary series related to Eqs. (22) and (16), namely

$$I_0(G) \approx (0)I_0 + (1)I_0G + \ldots$$

$$\approx \frac{1 + \lambda}{4} - \frac{\tau(1 + \lambda)^{3/2}}{4(1 + \lambda)} G + \ldots \quad (27)$$

$$H(\tilde{\epsilon} ; G) \approx \frac{(0)I_0^{3/2}}{1 + \tau} h(1 + (0)F')G + \ldots$$

$$\approx H(\tilde{\epsilon})G + \ldots$$

If these series are substituted into Eqs. (14) and (15), the lowest order solutions for $(0)I(\tilde{\epsilon})$ and $(0)F(\tilde{\epsilon})$ are given by these equations with the $H$-functions eliminated from the latter equations. Moreover, the boundary conditions of Eqs. (18)-(21) apply with $\tilde{\epsilon}$ and $\tilde{\epsilon}$ set to zero.

The next order solutions, those for $(1)I(\tilde{\epsilon})$ and $(1)F(\tilde{\epsilon})$, are given by the following equations after some rearrangement:

$$(1 + \tau \tilde{\epsilon} - 2(0)F)(1') - (h(0)F)^2[3(1 - 2\tilde{\epsilon}) + \kappa_2]$$

$$= -3(1') + [6(h(0)F)^2 - (1 - 2\tilde{\epsilon})2h(0)F(2h(0)F + 3h(0)F')]$$

$$+ 2\tau(\tau + (0)I') - 2\kappa_2 h(0)F(1 + (0)F')I(0)F - 2\tau h F H_1$$

$$- \frac{\tau}{h} (1') + [1 + \tau \tilde{\epsilon} + h(0)F(2(1 - 2\tilde{\epsilon}) + \tilde{\epsilon} + \lambda)](1')$$

$$= -I + [4h(0)F - 2(1 - 2\tilde{\epsilon})(h(0)F + h(0)F')]$$

$$- h(\tilde{\epsilon} + \lambda)(1 + (0)F')I(0)F - \frac{\tau}{h} H_1$$
Again expansions about \( \bar{c} = 0, 1 \) are called for; thus as \( \bar{c} \to 0 \) we take

\[
(1) I \approx (1) I_0 + (1) \beta_0 \bar{c} + \ldots
\]

\[
(1) F \approx (1) \gamma_0 \bar{c} + \ldots
\]

where \((1) I_0\) is known from Eq. (27). Expressions for \((1) \beta_0\) and \((1) \gamma_0\) are obtained by applying the expansion in \( G \) to Eqs. (19). The corresponding expansions about \( \bar{c} = 1 \) are similarly obtained from Eqs. (21). Thus again the solutions for \((1) I(\bar{c})\) and for \((1) F(\bar{c})\) are found by means of integration from the neighborhood of \( \bar{c} = 0, 1 \) with \((1) I_0\) and \((1) I_0 \) selected to achieve continuity at a common terminal point.

Finally, the influence of gravity on the spatial distributions of the variables is obtained from Eq. (17) with expansions in \( G \). We obtain

\[
\frac{x}{l_0} = \frac{(0) x}{l_0} + \frac{(1) x}{l_0} G + \ldots
\]

\[
= \frac{u_0}{u_{\infty} (1/0)^{1/2}} \int_{1/2}^{1} \frac{(1 + \tau \bar{c})^2 (1 + (0) F')}{(1 + \tau \bar{c})(1 - \bar{c})} d\bar{c}
\]

\[
+ \left[ \frac{u_0}{u_{\infty} (0/0)^{3/2}} \int_{1/2}^{1} \frac{(1 + \tau \bar{c})^2}{(1 + \tau \bar{c})(1 - \bar{c})} \left( (1) F' - \frac{1}{2} (1 + (0) F) \left( \frac{(1) I_0}{(0) I_0} \right) \right) d\bar{c} \right] G + \ldots
\]

We are now prepared to discuss the results of this analysis.

![FIGURE 2](image-url)  
**FIGURE 2** The variation of turbulent flame speed with the gravity parameter of \( G; \tau = 6 \).
3 NUMERICAL RESULTS

We confine our attention to $\tau = 6$, a value of the heat release parameter close to that of applied interest for most premixed turbulent flames. The parameters $\lambda$ and $\kappa_2$ are assigned the values $-0.133$ and zero respectively. The latter is one of the two values
used in earlier studies. The smaller value of $\lambda$ is dictated by the following considerations: The solutions require $F''(0) > 0$. Determination of the next terms, i.e., those proportional to $\tilde{\varepsilon}^2$, in Eqs. (18) leads to the requirement that for $G = 0$ $\lambda < 0$, a requirement confirmed by numerical experimentation. With these values the solutions for $G = 0$ are essentially those given in our earlier studies, in particular $I_0 = 0.217$ and the distributions of $(0)/(c)$ and $(0)F(c)$ are little changed.

In Figure 2 we show graphically the linear variation of the turbulent flame speed in the form $u_0/\tilde{u}_0$ with the gravity parameter $G$ given by Eq. (23). Extinction is indicated by either of two values depending on the criterion used: at $G = -2.17$ if $u_0 = 0$ at extinction or at $G = -1.09$ if $\gamma_0 = 0$ at extinction.

In Figures 3a–c we show the spatial distributions of the progress variable, the normalized intensity $\tilde{\alpha}u^2/\tilde{\alpha}u_0^2$ and the mean turbulent flux $F(\tilde{\varepsilon})$ for $G = 0, 0.2, -0.2$ as given by the linearized analysis. Note that the latter two sets of distributions for $G \neq 0$ deviate from those for $G = 0$ in both the ordinate and the abscissa. We see from Figure 3a that a flame thickness defined in terms of values of $\tilde{\varepsilon}$ close to zero and unity, e.g., $\tilde{\varepsilon} = 0.1$ and 0.9, is increased by positive values of $G$. The most significant quantitative influence of gravity is seen in Figure 3b which indicates a significant increase in the intensity of the velocity fluctuations throughout the flame and in the product stream for $G > 0$. This result is consistent with out intuitive expectation of the influence of buoyancy with unstable density distributions. Finally, Figure 3c suggests that there are countervailing influences of gravity on the mean flux; in the center of the flame $F$ is reduced for $G > 0$ but increased elsewhere in the flame. Note that since $\gamma_1 < 0$, gradient transport applies in the final approach to the product stream but that this occurs at distances beyond those shown in this figure.
4 CONCLUDING REMARKS

An analysis of the influence of gravity on premixed turbulent flames based on the Bray-Moss-Libby model of the aerothermochemistry of such flames and on a recently developed model for the mean rate of creation of product results in the following observations:

1. There is identified a gravity parameter $G \equiv g l_0 / u_t u_0$ which determines the extent of the influence of gravity on the properties of turbulent premixed flames. Horizontal flames propagating upward into cold reactants correspond to $G > 0$ and to unstable density distributions while those propagating downward correspond to $G < 0$ and to stable distributions. The quantities appearing in the definition of $G$ indicate that flames involving large turbulent scales in weak turbulence and in reactants with low laminar flame speeds are susceptible to gravitational effects.

2. To remove an ambiguity in the theoretical prediction of turbulent flame speed in general and of the influence of gravity thereon in particular we examine the implications of the Kolmogorov-Petrovsky-Piskunov theory and show that it leads to plausible results for that speed for both gravity-free flames and for the influence in question. Extinction is predicted at either of two negative values of $G$; one value corresponds to a zero flame speed while a larger value related to the onset of countergradient diffusion at the cold edge of the flame and thus to the possible loss of flameholding. The speed of upward propagating flames is predicted to increase indefinitely with increases in $G$. Calculation of a self-limiting flame speed requires modification of the model used for the mean rate of creation of product.

3. A linearized treatment of the influence of gravity indicates that the thickness of premixed turbulent flames and the intensity of the velocity fluctuations are increased by gravity, i.e., by $G > 0$. However, gravity affects only slightly the mean flux of the progress variable and thus of the various state variables.

4. Although physically plausible the absence of definite experimental data on gravitational effects on premixed turbulent flames precludes quantitative assessment of most of the predictions resulting from this analysis.

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