Velocity of Turbulent Flamelets with Realistic Fuel Expansion

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Nonlinear model equation for weakly turbulent premixed flames with realistic fuel expansion is proposed. The equation describes both the hydrodynamic flame instability and flame response to external turbulence in the regime of flamelets. On the basis of the proposed equation the velocity of turbulent flames is investigated for the cases of both weak and relatively strong turbulence. Comparison of the obtained turbulent velocity to experimental results shows that the model equation contains the essence of a more complete theory of turbulent flame speed.

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Turbulent flame velocity is one of the most important problems of combustion science, since it is a key parameter for numerical simulations of the burning process in engines. The notion of turbulent flames comprises several different regimes of burning with quite distinctive properties [1]. As far as burning in engines is concerned, flame typically propagates in the regime of turbulent flamelets. In this regime the turbulent flow is not so strong to influence internal flame structure, but strong enough to distort a flame front considerably on large length scales much larger than the flame thickness \( L_f \) [1]. Turbulent flamelets have been studied theoretically for a long time and many interesting results have been obtained; see, for example, Refs. [2–5]. However, most of these results were restricted to the limit of no fuel expansion, when the density of the burnt matter \( \rho_b \) is the same as the fuel density \( \rho_f \), with their ratio \( \Theta = \rho_f / \rho_b = 1 \). In this limit flame propagation does not influence external turbulent flow; the flame moves only with respect to the fuel just ahead of the flame front with the laminar flame velocity \( U_f \). In more elaborate models, variations of the laminar flame velocity, because of the flame stretch, may also be taken into account [6].

Still the limit of \( \Theta = 1\) is a rather peculiar one since fuel expansion for laboratory flames is typically as large as \( \Theta = 5–10 \). If one tries to incorporate the realistically large fuel expansion into turbulent models, then new effects come to play. First of all, one faces the intrinsic hydrodynamic flame instability obtained by Darrieus and Landau (the DL instability), which bends an initially planar flame front [1]. The instability growth rate is zero at \( \Theta = 1 \), and it increases with the increase of fuel expansion \( \Theta \). One can argue that the flame surface is almost randomly oriented with respect to the mean direction of propagation for strongly corrugated turbulent flames and the average effect of the DL instability goes to zero [7]. Many experiments support this reasoning; see [8] as a thorough review of the experimental results. However, at present there is mounting evidence that the role of the DL instability for turbulent flamelets may be quite noticeable at certain conditions [9,10]. Presumably, the role of the DL instability depends on the intensity of turbulent pulsations on the length scale comparable to the cutoff wavelength of the instability \( \lambda_c \), which is proportional to the flame thickness \( L_f \) [6]. Besides, even if the DL instability is not important for propagation of turbulent flamelets, the realistically large fuel expansion taken into account may considerably change the results of the turbulent flamelet theory [2,4,5].

Speaking of the DL instability one should remember that the nonlinear theory of the instability has also been restricted for a long time to the limit of small fuel expansion \( \Theta - 1 \ll 1 \) [11–13]. Quantitative theoretical description of strongly curved flames with realistically large fuel expansion \( \Theta \) resulting from the DL instability has been developed only recently [14–16]. Attempts to incorporate both turbulence and the DL instability into numerical models have been limited to the case of weak fuel expansion \( \Theta - 1 \ll 1 \) [10].

In the present paper we propose the following model nonlinear equation for the flame position \( z = F(x,y,t) \) that incorporates both the DL instability and weak external turbulence for flamelets with arbitrary fuel expansion including realistically large values \( \Theta = 5–10 \):

\[
\frac{\Theta + 1}{2\Theta} (1 + C_1 L_f \hat{\Phi}) \frac{\partial^2 F}{\partial t^2} + (1 + C_2 L_f \hat{\Phi}) \frac{1}{U_f^2} \frac{\partial F}{\partial t} + \frac{(\Theta - 1)^3}{16\Theta} [(\nabla F)^2 - (\hat{\Phi} F)^2] - \frac{\Theta - 1}{2} \left( 1 - \frac{\lambda_c}{2\pi} \hat{\Phi} \right) \hat{\Phi} F - \left( 1 + \frac{\hat{\Phi}^{-1}}{U_f} \frac{\partial}{\partial t} \frac{u_T}{U_f} \right) = 0,
\]

where the operator \( \hat{\Phi} \) implies multiplication by absolute value of the wave number in Fourier space

\[
\hat{\Phi} F = \frac{1}{4\pi^2} \int |k| F_k \exp(ik \cdot x) dk.
\]

The model equation (1) is written in the reference frame of the average position of the turbulent flame front, so that \( U_w \) stands for the average velocity of the turbulent flamelet. The model equation comprises three original
theories developed rigorously in the previous papers: (1) the linear theory of the DL instability by Pelce and Clavin [6]; (2) the nonlinear theory of curved flames by Bychkov [14]; and (3) the linear theory of flame dynamics in a turbulent flow by Seary and Clavin [3]. Without the last term the model equation describes development of the DL instability at linear and nonlinear stages. Together with the last term the equation determines (at least qualitatively) flame response to external turbulence, where \( u_T \) is \( z \) component of the turbulent velocity. The numerical factors \( C_1 \) and \( C_2 \) in Eq. (1) depend on fuel expansion \( \Theta \) and other internal flame parameters according to [17]. Particularly, in the case of constant thermal conduction and unit Lewis number one has \( C_1 = 0, C_2 = \Theta \ln \Theta / (\Theta - 1) \). Similar to [10] one can take the following model for the turbulent term:

\[
u_T = \sum U_i \cos(k_i x + \varphi_{ix}) \cos(k_i y + \varphi_{iy}) \times \cos(k_i x + \varphi_{ix}) \cos(k_i y + \varphi_{iy}),
\]

where \( \varphi_{ix}, \varphi_{iy}, \varphi_{ix}, \varphi_{iy} \) are random phases. The amplitudes \( U_i \) are determined by the Kolmogorov spectrum \( U_i \propto k_i^{5/6} \) and the rms-turbulent velocity in one direction in this model is given by the formula

\[
U_{\text{rms}}^2 = \sum U_i^2 / 8.
\]

Concerning the dependence of the turbulent frequency on the turbulent wave number, the dimensional analysis suggests two possibilities [10]: \( \Omega_i \propto k_i^{1/6} \) and \( \Omega_i \propto k_i^{2/3} \). One more possibility comes from the Taylor hypothesis used in a large number of papers on turbulent flames (see, for example, Refs. [2,5,7]). With the hypothesis, “the time integral of the longitudinal derivative of a fluctuating quantity is approximately \( 1/U_w \) times the fluctuating quantity” [1], which implies \( \Omega_i \ll U_w k_i \) in the scope of the model (3).

In general, the solution to the model equation (1) with a forcing term (3) involves both induced turbulence and the intrinsic DL instability of the flame front. However, experiments [8] point out that the instability is often of small importance. In this sense it is interesting to consider a particular solution to Eq. (1) related to the external turbulence only. Keeping in mind the validity conditions of Eq. (1) we start with the case of weak turbulence \( U_w - U_f \ll U_f \). Besides, since integral turbulent length scale \( L_T \) is typically much larger than the flame thickness \( L_f \), we can consider an infinitely thin flame front neglecting the dependence of the turbulent flame velocity on the ratio \( L_T/L_f \) similar to [5]. In the case of weak turbulence and weak nonlinearity the solution to Eq. (1) takes the form

\[
F = \sum \cos(k_i x + \varphi_{ix}) \cos(k_i y + \varphi_{iy}) \times [F_{ci} \cos(\omega_i t + \varphi_{iz}) + F_{si} \sin(\omega_i t + \varphi_{iz})],
\]

where \( \omega_i = U_w k_i \pm \Omega_i \). Substituting (5) into (1) we find with the accuracy of the linear terms the set of equations for \( F_{ci}, F_{si} \)

\[
- \left( \Theta + \frac{1}{2\Theta} \omega_i^2 + \frac{\Theta - 1}{2} 2U_w^2 k_i^2 \right) F_{ci} + \frac{\sqrt{2} \omega_i U_w k_i F_{si}}{\omega_i U_w k_i} = \frac{\sqrt{2} U_w k_i}{\omega_i U_w k_i}.
\]

Introducing designations \( A_i = (\Theta + 1)\omega_i^2 / 2\Theta + (\Theta - 1)U_w^2 k_i^2 \) and \( B_i = \sqrt{2} \omega_i U_w k_i \), we find expressions for the amplitudes \( F_{ci}, F_{si} \)

\[
F_{ci} = \frac{A_i \omega_i + \sqrt{2} B_i U_w k_i}{A_i^2 + B_i^2} U_w,
\]

\[
F_{si} = \frac{B_i \omega_i - \sqrt{2} A_i U_w k_i}{A_i^2 + B_i^2} U_w.
\]

The average turbulent flame velocity is related to nonlinear terms of Eq. (1)

\[
\frac{U_w}{U_f} - 1 = \Theta \frac{1}{2} \langle (\nabla F)^2 \rangle + \frac{(\Theta - 1)^3}{16\Theta} \langle (\nabla F)^2 - \langle F \rangle^2 \rangle.
\]

where \( \langle \ldots \rangle \) denotes time and space averaging. One can check that the last two terms in the right-hand side of Eq. (10) give zero after averaging. Then substituting representation Eq. (5) with the amplitudes (8) and (9) into Eq. (10) we find

\[
\frac{U_w}{U_f} - 1 = \Theta \frac{1}{2} \langle (\nabla F)^2 \rangle = \Theta \sum \frac{k_i^2}{8} (F_{ci}^2 + F_{si}^2),
\]

or

\[
\frac{U_w}{U_f} - 1 = \Theta \sum \frac{U_w^2 k_i^2} {8} \frac{\omega_i^2 + 2U_w^2 k_i^2} {A_i^2 + B_i^2}.
\]

Keeping in mind the definitions for \( A_i \) and \( B_i \), one can check that for \( \Theta = 1 \) the velocity increase becomes

\[
\frac{U_w}{U_f} - 1 = \frac{U_w^2 k_i^2} {8\omega_i^2}.
\]

This expression coincides with the well-known Clavin-Williams formula [2] written for the turbulence model (3). With the help of the Taylor hypothesis for weak turbulence \( U_w - U_f \ll U_f \) Eq. (13) takes the form

\[
\frac{U_w}{U_f} - 1 = \frac{1}{U_f^2} \sum \frac{U_w^2} {8} = U_{\text{rms}}^2 / U_f^2.
\]

Using the Taylor hypothesis in the case of arbitrary fuel
Because of the narrow width of every band one can write
\[ U_w/U_f - 1 = C_\Theta^2 \frac{U_{\text{rms}}^2}{U_f^2} \]  
(15)
with the coefficient \( C_\Theta \) depending on fuel expansion as
\[ C_\Theta^2 = \frac{12 \Theta^3}{(2 \Theta^2 - 1 + 1)^2 + 8 \Theta^2} \]  
(16)
The coefficient \( C_\Theta \) varies not so strongly with variations of fuel expansion being somewhat lower than unity in the case of realistic fuel expansion \( \Theta = 5-10 \). For example, for the fuel expansion \( \Theta = 5-6 \) characterizing propane flames the coefficient is \( C_\Theta = 0.7-0.8 \). In the case of 2D turbulence Eq. (15) goes over to
\[ U_w/U_f - 1 = \frac{C_{\Theta,2D}^2}{2} \frac{U_{\text{rms}}^2}{U_f^2} \]  
(17)
with the correction coefficient
\[ C_{\Theta,2D}^2 = \frac{8 \Theta^3}{(\Theta^2 + 1)^2 + 4 \Theta^2} \]  
(18)
The way to go over from the case of weak turbulence to the case of strongly turbulent flamelets has been pointed out by Yakhot [5]. Yakhot proposed to split the turbulent velocity field into narrow, almost monochromatic bands in the wave-number space with small amplitude for each band. Then turbulent flame velocity comes as an integral action of all bands. The first band provides the increase of flame velocity given by Eq. (12)
\[ U_{w1} - U_f = U_f d_1 \Sigma, \]  
(19)
where \( d_1 \Sigma \) denotes the sum
\[ d_1 \Sigma = \Theta \sum \frac{U_i^2 k_i^2 \omega_i^2 + 2 U_f^2 k_i^2}{A_i^2 + B_i^2} \]  
(20)
corresponding to the first band. The second band provides similar velocity amplification, but with \( U_{w1} \) playing the role of flame velocity
\[ U_{w2} - U_{w1} = U_{w1} d_2 \Sigma. \]  
(21)
Because of the narrow width of every band one can write a continuous equation
\[ dU_w = U_w d\Sigma, \]  
(22)
which can easily be solved as \( U_w = U_f \exp(\Sigma) \), or
\[ U_w = U_f \exp\left(\Theta \sum \frac{U_i^2 k_i^2 \omega_i^2 + 2 U_f^2 k_i^2}{A_i^2 + B_i^2}\right) \]  
(23)
with the sum taken over the whole turbulent spectrum. Using the Taylor hypothesis we can reduce the above equation to
\[ \ln(U_w/U_f) = f_\Theta \frac{U_{\text{rms}}^2}{U_w^2}, \]  
(24)
where the new factor \( f_\Theta \) depends not only on the fuel expansion \( \Theta \), but also on the ratio of the turbulent flame velocity to the laminar velocity \( U_w/U_f \) as
approximately through the middle of the “cloud” of experimental points, which demonstrates good agreement of the experimental results and the analytical formula (24) of the present paper. Because of this good agreement we may expect that the model equation (1) provides not only qualitative but also quantitative description of turbulent flames with realistic fuel expansion. To some extent, scattering of the experimental results may be explained by different fuel expansion for different flames used in experiments. Besides, different experimental points have been observed for different integral turbulent length scales $L_T/L_f$. Dependence of turbulent flame velocity on $L_T/L_f$ may also be expressed in the form of dependence on the Reynolds number of the turbulent flow, on the Damkohler number characterizing the turbulent time $L_T/U_{\text{rms}}$ versus the reaction time $L_f/U_f$, or on the Karlovitz number characterizing the rate of flame stretch versus the reaction rate. It has been shown in [7] that in a sufficiently strong turbulent flow the flamelet regime of flame propagation breaks down and a turbulent flame takes the form of a thick flame with internal structure affected by turbulence. In this case instead of an almost linear increase of turbulent flame velocity $U_w/U_f$ with an increase of $U_{\text{rms}}/U_f$ demonstrated by theoretical curves in Fig. 1 one has saturation of the value $U_w/U_f$. The saturation comes earlier at smaller values of the Reynolds number of the turbulent flow. Particularly, some experimental points of the lower part of the cloud may be explained by this effect, since they have been observed for larger values of the Karlovitz number (smaller values of the Reynolds number). In the case of relatively large Karlovitz number one may also expect nonzero influence of terms related to finite flame thickness in the model equation (1): These terms have been neglected in the derivation of the analytical formula (24).

Finally, the analytical formula (24) does not take into account direct influence of the DL instability. At the same time, if we consider a flame front in a shear flow characterized by a length scale $\lambda$, then the model equation (1) has a bifurcation point when the length scale of the shear flow coincides with the cutoff wavelength of the instability $\lambda = \lambda_c$. The same effect is expected in the case of weak turbulence. From this point of view the solution leading to Eq. (24) is only one of different possible solutions to Eq. (1). Even more, other solutions may be characterized by larger flame velocity, since the DL instability leads to additional bending of the flame front. Particularly, it was observed in numerical simulations [10] that joint effect of the DL instability and external turbulence leads to larger fractal dimension of the flame front compared to the fractal dimension produced by turbulence alone. Presumably, some experimental points in Fig. 1 above the theoretical curves of the present paper may be explained by direct influence of the DL instability. The solution to Eq. (1) taking into account the DL instability requires much more space and will be presented elsewhere.

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