The Quenching of Deflagration Waves

J. BUCKMASTER
Mathematics Department and Department of Theoretical and Applied Mechanics,
University of Illinois, Urbana, Illinois 61801

A simple model of a low Mach number deflagration wave is studied, with emphasis on the effects of a phenomenological heat loss term. It is shown by explicit construction that two quite different solutions, a fast wave and a slow wave, are possible in the adiabatic limit. In both cases the chemical reaction goes to completion and all the fuel is consumed, but for the fast wave the temperature increases from the ambient value to the usual adiabatic flame temperature, whereas the slow wave is isothermal. An analysis for finite heat loss is carried out in the realistic limit of infinite activation energy. In general this leads to a much simplified numerical problem, but for values of the flame speed of greatest physical interest an analytical description is possible. This predicts quenching when the heat loss is too great or the reaction rate too small, and yields simple explicit quenching criteria.

1. Introduction

It is well known that a laminar diffusion flame can not propagate in a mixture confined within too narrow tubes, a result generally attributed to heat losses through the walls. The problem has been widely discussed starting with the pioneering work of Spalding [7], who describes an approximate analysis. Numerical computations by Adler and Spalding [1] and others (e.g., Gerstein and Stine [3]) confirm the qualitative predictions of the earlier study, the most important of which are that if the heat loss is too great the flame is quenched and there is no steady-state solution; and for heat losses less than the quenching limit there are two solutions, a fast wave and a slow wave. Thus the response diagram sketched in Fig. 2 is characteristic of the problem. The fast wave is apparently stable and is the one usually observed. Spalding and Yumlu [8] reported observations of the slow wave in a specially constructed stabilizing apparatus, and although questions were raised about the accuracy of their heat loss measurements, Yumlu [9] sustained the view that the slow wave is observable. Nevertheless, the slow flame is not of great physical interest.

In view of the long and illuminating history of this problem, the reader may wonder why yet another analysis is required, particularly since as we shall see, few new qualitative insights are forthcoming. What distinguishes the analysis of the present paper from earlier analytical work is that after a clearly defined set of model equations is written down, their solution is discussed in a rational way without recourse to ad hoc approximations. One of the tiroblems with a complicated subject such as combustion is that even after simplified model equations have been postulated, analytical results are usually only possible after a great many, often questionable, additional approximations have been adopted. One then has the challenge of deciding whether the results are creatures of the model equations or of the analysis. Small wonder, then, that importance is attached to numerical confirmation of any predictions. The present analysis, being rational, is an accurate substitute for the numerical work and indeed we essentially duplicate most of the results of Gerstein and Stine [3]. Of course our results have all the familiar advantages that a simple formula has over numerical output. Moreover, the present work lays the foundation for
analyses of model equations more general than those adopted here. Suppose for example, one wanted to simulate the effect of wall capture of hot radicals (Simon, Belles, and Spakowski [6]). This could be done by adding an appropriate sink term to the species equation and then carrying out an analysis similar to the present one.

The key to the present analysis is the adoption of Arrhenius kinetics characterized by an activation temperature that is much larger than the adiabatic flame temperature. An asymptotic analysis is then possible. Activation energy asymptotics has only recently been exploited to any significant extent, and has proven to be an extremely powerful tool in a variety of problems. For example, Bush and Fendell (1970) solved the problem of adiabatic deflagration waves in this way, thereby rendering obsolete a large amount of earlier work in the subject. We perform the same service for the non-adiabatic case.

The problem is discussed in several stages. After the formulation in §2 the fast-and slow-flame adiabatic limits are described in §3. The fast-flame limit can only be analyzed when the activation energy is large, and this is Bush and Fendell’s problem, but the slow-flame limit can be discussed without any such restriction. This analysis brings out the striking physical differences between slow flames and fast flames.

Section 4 is devoted to activation energy asymptotics for the special case of a linear dependence of the heat loss on temperature. Only a brief outline of the analysis is given since it is only needed if a complete description of the slow-flame branch is required, and furthermore, analytical results are not possible. The analysis of the fast-flame branch is carried out in §5 when the heat loss is an arbitrary function of temperature. This section contains the major results of the paper, including explicit quenching criteria.

2. Formulation

Williams [10] formulates a mathematical problem for adiabatic deflagration waves and we shall follow the formulation closely with some generalizations, the most important of which is the addition of a heat loss term to the energy equation. A one-step reaction mixture → product is assumed so that the combustion field is characterized by two variables, $T$ the temperature and $Y$, the product mass-fraction. The characteristic Mach number of the flame is assumed sufficiently small for the pressure to be sensibly constant. A constant specific heat $C_p$ is assumed, but the heat conduction coefficient $\lambda$ is allowed to vary with temperature. This makes it expedient, following Williams, to define an independent space variable by

$$\zeta = mc_p \int^x \frac{dx}{\lambda},$$

where $m$, the mass flux through the flame as observed in a frame in which the flame is stationary, is a constant that must be determined. Then

$$\frac{dY}{d\zeta} = Le \left[ Y - \epsilon \right], \quad (2.1)$$

where $Le$ is the Lewis number, assumed constant, and $\epsilon$ is the product flux fraction as defined by Williams. Changes in $\epsilon$ are governed by

$$\frac{d\epsilon}{dx} = \frac{\omega_2}{m},$$

where $\omega_2$ is the mass of product generated per unit volume per unit time, so that we shall write

$$\frac{d\epsilon}{d\zeta} = \frac{B(T)}{m^2} \left( 1 - Y \right) \exp \left( \frac{E_1}{R^o \tilde{T}} - \frac{E_1}{R^o T} \right), \quad (2.2)$$

Where $\tilde{T}$ is the adiabatic flame temperature defined by

$$\tilde{T} = T_w + \frac{(h_m^o - h_p^o)}{C_p}.$$

In this expression $T_w$ is the ambient temperature, $h_m^o$ is the heat of formation of the mixture and $h_p^o$ is the heat of formation of the
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Product. Defining a new temperature variable by

\[ T = T_{\infty} + \left( \frac{h_m^o - h_p^o}{C_p} \right) \tau, \]

Eq. (2.2) may be written in the form

\[ \frac{d\epsilon}{d\xi} = \theta^2 \Omega \chi(\tau) (1 - Y) \exp\left[ \frac{\theta(\tau-1)}{\alpha^+ \tau} \right], \quad (2.3) \]

where \( \theta \) is a non-dimensional activation energy defined by

\[ \theta = \frac{E_1}{R_0 T}, \]

and

\[ \alpha = \frac{(h_m^o - h_p^o)}{C_p T_{\infty}}. \]

\( \Omega \) is a constant that is a measure of the reaction rate divided by \( m^2 \), and in recognition of this we define \( \omega' \) by

\[ \Omega = \frac{\omega'}{m^2}. \]

is an arbitrary function that accounts for the preexponential temperature dependence of the reaction rate and also includes a contribution from \( \lambda(T) \).

The energy equation may be written in the form

\[ \frac{d^2 \tau}{d\xi^2} = \frac{d\epsilon}{d\xi} - \frac{d\epsilon}{d\xi} + K \varphi(\tau), \quad (2.4) \]

where

\[ K \varphi(\tau) = \frac{\lambda Q}{m^2 C_p (h_m^o - h_p^o)}, \]

and \( Q \) is the heat loss per unit volume per unit time. \( \varphi(\tau) \) is an arbitrary function (except for the restriction \( \varphi(0) = (0) \) and thus permits an arbitrary dependence of \( Q \) on \( \tau \). Note that the constant \( K \) is a measure of the heat loss divided by \( m^2 \) and in recognition of this we define \( Q' \) by

\[ K = \frac{Q'}{m^2}. \]

Eliminating \( \epsilon \) leads to the equations that we shall solve in this paper, namely

\[ \frac{1}{Le} \frac{d^2 Y}{d\xi^2} = \frac{dY}{d\xi} - \theta^2 \Omega \chi(\tau) (1 - Y) \exp\left[ \frac{\theta(\tau-1)}{\alpha^+ \tau} \right], \quad (2.5a) \]

and

\[ \frac{d^2 \tau}{d\xi^2} = \frac{d\tau}{d\xi} + K \varphi(\tau) - \theta^2 \Omega \chi(\tau) (1 - Y) \exp\left[ \frac{\theta(\tau-1)}{\alpha^+ \tau} \right], \quad (2.5b) \]

Boundary conditions appropriate for this system are

\[ \xi \rightarrow -\infty \quad \tau \rightarrow 0, \quad Y \rightarrow 0, \quad (2.6a) \]

\[ \xi \rightarrow +\infty \quad (i) \quad K \neq 0: \tau \rightarrow 0, \quad \frac{dY}{d\xi} \rightarrow 0, \quad (2.6b) \]

\[ (ii) \quad K = 0: \tau \rightarrow 1, \quad Y \rightarrow 1. \quad (2.6c) \]

It is apparent that to resolve the cold boundary difficulty it is necessary to choose either \( \alpha^+ = 0 \) or \( \chi(0) = 0 \). Then the reaction rate vanishes as \( |\xi| \rightarrow \infty \) when \( K \neq 0 \), since \( \tau \) then vanishes far from the flame. One consequence of this is that when there are heat losses we cannot infer that \( Y \rightarrow 1 \) as \( \xi \rightarrow +\infty \). Indeed, as the subsequent analysis will reveal, there is in general a residue of unburnt mixture left behind by the flame.

This conclusion is different from that of Gerstein and Stine [3] simply because they assume a reaction rate proportional to

\[ (1 - Y) \exp\left( -\frac{E_1}{R_0 T} \right), \]
and \( T_\infty \neq 0 \) in their calculations so that the only way this expression can vanish as \( \xi \to +\infty \) is if \( Y \to 1 \). The present formulation, which permits the indefinite existence of a cold unburnt mixture, is clearly more realistic.

It is well known (Johnson [4]) that mere resolution of the cold boundary difficulty does not guarantee a unique flame speed for the adiabatic problem; in general there is a continuum of possible flame speeds. However, by introducing a cut-off temperature, slightly larger than the cold-boundary temperature, and below which the reaction rate is zero, uniqueness is assured [4]. Similar difficulties might occur for the non-adiabatic problem but can presumably be handled by a suitable choice of the behavior of \( \chi(\tau) \) near \( \tau = 0 \) (cut-off corresponds to \( \chi = 0 \) when \( \tau < \tau^* \), where \( 0 < \tau^* < 1 \)). Explicit choices do not have to be made when the activation energy is large, since the solution is sensitive only to the behavior of the reaction rate near the temperature maximum.

We shall analyze the system (2.5)-(2.6) by supposing that \( K, \alpha, \theta \) and \( Le \) are specified constants, \( \varphi \) and \( \chi \) are prescribed functions, and \( \Omega \) is a constant that must be determined as part of the solution. Thus \( \Omega \) is the basic eigenvalue of the problem.

3. Adiabatic Limits

Adiabatic deflagration corresponds to \( Q' = 0 \), i.e.,

\[ Km^2 = 0. \]  

(3.1)

The classical adiabatic limit is attained when \( K = 0 \) and this is the problem analyzed by Bush and Fendell [2] in the limit \( \theta \to \infty \). Their analysis requires consideration of the combustion field in three distinct regions. In \( \xi < 0 \) the chemistry is frozen; in \( \xi > 0 \), chemical equilibrium prevails\(^1\) so that \( Y = 1 \) correct to all algebraic order in \( \theta \); between these two regions there is a flame-sheet which smooths out the gradients and in which appropriate variables are \( \eta = \theta^2 \xi, \gamma(\eta) = \theta(\eta-1), \tau(\eta) = \theta(\eta-1) \). Solutions in these three regions are easily constructed and matched with each other and in this way the following expression for the flame-sheet is deduced,

\[ m = [2\omega' \chi(1) Le (1 + \alpha^{-1})^2]^{1/2}. \]  

(3.2)

The limit corresponds to \( m \to \text{constant}, Q' \to 0 \) when \( \omega' \) is fixed, or \( m \to \infty, \omega' \to 0 \) when \( Q' \) is fixed. The temperature profile is sketched in Fig. 1.

An alternative adiabatic limit comes from the choice \( K \to \infty \) for this corresponds to \( m \to 0 \) in such a way that, in the limit, Eq. (3.1) is satisfied. Crude ad hoc analyses (e.g., von Karman, see Williams [10]) have indicated something of the nature of the solution in this limit, but a rational asymptotic description is possible for large \( K \). We shall sketch the details for the special case

\[ \varphi \equiv \tau, \quad \chi \equiv \tau^\gamma, \quad \alpha = \infty. \]  

(3.3)

It is appropriate to write

\[ \theta^2 \exp(\theta) \Omega = K^2 \exp(\frac{\theta}{\tau^*}), \]  

(3.4)

where \( \tau^*(K) \) is the maximum value of \( \tau \), assumed located at the origin, and \( \Omega \) is an \( O(1) \) quantity.

On a scale \( \xi = O(1) \) the reduced equations, correct to all algebraic orders in \( K \), are

\[ \tau = 0 \]  

(3.5a)

and a solution for \( Y \) is sought in the form

\[ Y^\pm \sim Y^\pm_0 + \frac{1}{\sqrt{K}} Y^\pm_1 + O(\frac{1}{K}), \]  

(3.5b)

where the superscript \((+\) refers to \( \xi > 0, (-\) to \( \xi < 0 \). To avoid subsequent contradictions in the analysis we are compelled to make the choice

\[ Y^+_0 = 1. \]

Moreover, \( Y^+_1 = B, \) a constant.

On a scale \( \xi = O(\frac{1}{\sqrt{K}}) \), appropriate variables are

\(^1\)I.e., if fuel were present it would be instantly consumed.
\[ \eta = \sqrt{K} \xi, \quad t(\eta) = \sqrt{K} \tau, \quad t_\pi = \sqrt{K} \tau_\pi, \]
\[ y(\eta) = \sqrt{K} \left( Y - 1 \right), \] (3.6)

and the equations, correct to all algebraic orders in \( K \) when \( \eta = O(1) \) are
\[ \frac{1}{Le} \frac{d^2 y}{d\eta^2} = \frac{1}{\sqrt{K}} \frac{dy}{d\eta}, \]
\[ \frac{d^2 t}{d\eta^2} = t + \frac{1}{\sqrt{K}} \frac{dt}{d\eta}. \]

Solutions are sought in the forms
\[ y_0^\pm \sim y_0^0 + \frac{1}{\sqrt{K}} y_1^\pm + O\left( \frac{1}{K} \right), \]
\[ t_0^\pm \sim t_0^0 + \frac{1}{\sqrt{K}} t_1^\pm + O\left( \frac{1}{K} \right), \]

and amongst other results,
\[ y_0^+ = B. \]

Finally, an inner region (a flame-sheet) is described on a scale \( \xi = O\left( \frac{1}{K} \right) \) in terms of the variables
\[ \xi = \sqrt{K} \eta, \quad s(\xi) = \sqrt{K} \left( t - t_\pi \right), \quad z(\xi) = \sqrt{K} \left( y - B \right), \]
\[ \Omega = \frac{2^2 + \gamma}{Le} K \frac{3}{2} + \frac{\gamma}{2} \exp\left[ 2\sqrt{K} \right] + Le - 1. \] (3.10)
Clearly the limit corresponds to $m \to 0, Q' \to 0$ if $\omega'$ is fixed, or $m \to 0, \omega' \to \infty$ if $Q'$ is fixed. Note that $B$ is a measure of the amount of unburnt mixture left behind by the flame. This residue is $O[1/\sqrt{K}]$ and vanishes in the limit.

The results of this section are consistent with the familiar response diagram, as sketched in

**Fig. 2. Flame speed as a function of heat loss, $\omega$ fixed.**

**Fig. 3. Flame speed as a function of reaction rate, $E$ fixed.**
Figs. 2 and 3. The points A correspond to the result (3.2) and the points B to (3.10).

Figure 4 is a sketch of the flame structure in the slow-flame adiabatic limit. The reaction goes to completion in the limit and the profile for \( Y \) is indistinguishable, on the \( \xi \) scale, from that in the fast-flame limit outside of the flame sheet. However, in striking contrast to the fast-flame limit, the structure is isothermal. Of course, on a physical scale the slow flame is much thicker than the fast flame.

\[
\frac{1}{\text{Le}} \frac{d^2 Y}{d\xi^2} = \frac{dY}{d\xi} - \frac{\Lambda}{\delta^2} Y^2 (1 - Y) \exp\left[\frac{\theta}{\tau*} (1 - \tau)\right],
\]

and except in the neighborhood of the origin the combustion is frozen in the limit \( \theta \to \infty \). In the frozen regions a solution is sought in the form

\[
Y \sim Y_{*0}^+ + \delta Y_{*1}^+ + O(\delta^2)
\]

\[
\tau \sim \tau_{*0}^+ + \delta \tau_{*1}^+ + O(\delta^2).
\]

There is a flame-sheet of thickness \( O(\delta) \) centered at \( \xi = 0 \) and the appropriate expansions within this sheet are

\[
\xi = \delta \eta,
\]

\[
Y \sim 1 + \delta y \eta + O(\delta^2),
\]

\[
\tau \sim \tau* + \delta \tau 1 + O(\delta^2),
\]

\[
\Lambda \sim \Lambda 0 + \delta \Lambda 1 + O(\delta^2).
\]

Matching with the solution in the frozen regions leads to the following problem for the flame-sheet structure:

\[
\frac{d^2 \tau_1}{d\eta^2} \sim \Omega 0 \text{Le} \tau* 4 \exp\left[\frac{\theta}{\tau*} \tau 1\right] + \tau* 1
\]
\[ t_1(0) = t_1'(0) = 0, \quad (4.2b) \]
\[ t_1'(\infty) = \frac{-1 - \sqrt{1 + 4K}}{2\sqrt{1 + 4K}} \equiv \mu^+, \quad (4.2c) \]
\[ t_1'(-\infty) = \frac{1 + \sqrt{1 + 4K}}{2\sqrt{1 + 4K}} \equiv \mu^-, \quad (4.2d) \]
\[ t \sim \mu^\pm \eta + \rho^\pm + O(1) \quad \text{as} \quad \eta \to \pm \infty, \quad (4.2e) \]

where
\[ \tau_{*0} = (1 + 4K)^{-\frac{1}{2}} \quad (4.2f) \]

and
\[ \tau_{*1} (1 + 4K)^{\frac{1}{2}} = -Le W + \rho^- \left[ -\frac{1}{2} - \frac{1}{2} \sqrt{1 + 4K} \right] \]
\[ - \rho^+ \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4K} - Le \right]. \quad (4.2g) \]

Since four boundary conditions are given for the second order equation its solution implies the simultaneous determination of \( W(K), \rho^\pm(K) \) and
\[ Le \tau_{*0}^4 + \gamma \Omega_0 \exp(-\frac{\theta}{\tau_{*0}} + \tau_{*1}) \equiv f(K). \quad (4.3) \]

is then known from (4.2g) as a function of \( K \) and the Lewis number, and finally (4.3) determines \( \Omega_0 \) as a function of \( K, \theta, Le, \) and \( \gamma \). In principle then, the problem is completely solved once the four functions \( W, \rho^\pm \) and \( f \) have been determined.

It is important to note that
\[ \Omega \sim e^{\theta} \left( \frac{\lambda_0 + \delta \lambda_1 + \ldots}{\tau_{*0}^4 + \delta \tau_{*1} + \ldots} \right) \]

(\text{thus} \( \Omega_0 = \frac{\lambda_0}{\tau_{*0}^4} \exp\left[\frac{\theta}{\tau_{*0}} - \theta - \tau_{*1}\right] \)),

and the variation of \( \Omega \) with \( K \) is essentially determined by the factor \( \exp(\theta/\tau_{*0}) \). It would be inconsistent in computing the derivative of \( \Omega \) to include the lower order terms that arise by differentiating \( \tau_{*1} \) or \( \frac{\lambda_0}{\tau_{*0}^4} \) unless the term proportional to it is also retained. Now
\[ \frac{d^n \Omega}{dK^n} \sim \left[ \frac{2\theta}{(1 + 4K)^{\frac{1}{2}}} \right]^n \Omega \]

and since \( \Omega \) is positive it follows the \( \Omega \) and all its derivatives are monotone, decreasing functions of \( K \). Furthermore, as \( K \to 0 \) we have from (3.2)
\[ \Omega_0 \to \frac{1}{2 Le}. \]

whereas (3.10) shows that
\[ \frac{3}{2} + \frac{\gamma}{2} \exp\left(2\theta \sqrt{K}\right) \quad \text{as} \quad K \to \infty. \]

Thus if \( Q', \theta, Le \) and \( \gamma \) are fixed the graph of \( \Omega_0 vs m \) has the form shown in Fig. 5. Where it intersects the graph of \( \omega'/m^2 \) it defines two possible flame speeds. Clearly if \( Q' \) is too large or \( \omega' \) is too small the curves do not intersect. Thus, provided the problem (4.2) has a solution it is apparent that it will generate response diagrams of the form sketched in Figs. 2 and 3.

Explicit construction of the response requires numerical computations which we have not performed since \( O(1) \) values of \( K \) correspond to the slow-flame branch.1 The fast-flame branch corresponds to \( O(1/\theta) \) values of \( K \) and then the problem can be solved without recourse to the computer. This analysis could continue from (4.2) but the results would then only be valid for the special choices (3.3). To solve the general problem it is necessary to start with the choice \( K = O(1/\theta) \) a priori, and this the point of departure of § 5.

In § 3 it was found that the large \( K \) there is a residue of unburnt mixture behind the flame. The present analysis for finite \( K \) reveals that the flame speed is exponentially smaller than (3.2).

1More precisely, to that part of the response for which the flame speed is exponentially smaller than (3.2).
The boundary condition $\tau(\infty) = 0$ arises solely because of heat losses, since in the adiabatic case the temperature does not decrease behind the flame front and the downstream boundary condition is then $\tau(\infty) = 1$. Small heat losses cause only a weak decay in temperature behind the wave and significant deviation from $\tau = 1$ occur only on the large scale $\xi = O(\theta)$, $\xi > O$. We shall not consider the structure on this scale so that the solution that we shall describe in $\xi > 0$ is a small perturbation about $\tau = 1$.

Limiting equations valid when $0 \sim o(1)$ may now be written down. When $\xi$ is negative and $O(1)$, $(r - 1)$ is a negative $O(1)$ quantity so that the reaction term is transcendentally small in the limit. Then to all algebraic orders the equations

$$
\frac{d^2 Y}{d\xi^2} = dY \frac{d\xi}{d\xi},
$$

so that there is $O(1)$ residue. Both $W$ and $\rho^+$ Vanish as $K \to 0$.

5. Activation Energy Asymptotics, $K = O(1)$

In this section the problem posed by (2.5), (2.6) is analysed in the limit $\theta \to \infty$ when $K = O(1/\theta)$. It is helpful to write

$$
K = -\frac{\tilde{K}}{\theta},
$$

Where $\tilde{K}$ is $O(1)$, and then the analysis is relatively straightforward once certain key facts have been established.

It is clear from the boundary conditions that $\tau$ assumes a maximum, and with no loss of generality this maximum may be located at $\xi = 0$. In the absence of heat losses ($K = O)$ the maximum temperature is the adiabatic flame temperature ($\tau = 1$) so that it is not surprising that in the general case for which the heat losses are $O(1/\theta)$, the maximum value of $\tau$ differs from 1 by $O(1/\theta)$. This is the motivation for writing the chemical production term in the in the form (2.2).

The boundary condition $\tau(\infty) = 0$ arises solely because of heat losses, since in the adiabatic case the temperature does not decrease behind the flame front and the downstream boundary condition is then $\tau(\infty) = 1$. Small heat losses cause only a weak decay in temperature behind the wave and significant deviation from $\tau = 1$ occur only on the large scale $\xi = O(\theta)$, $\xi > O$. We shall not consider the structure on this scale so that the solution that we shall describe in $\xi > 0$ is a small perturbation about $\tau = 1$.

Limiting equations valid when $\theta \to \infty$ may now be written down. When $\xi$ is negative and $O(1), (\tau - 1)$ is a negative $O(1)$ quantity so that the reaction term is transcendentally small in the limit. Then to all algebraic orders the equations

$$
\begin{align*}
\frac{d^2 \tau}{d\xi^2} &= \frac{d\tau}{d\xi} + \frac{\tilde{K}}{\theta} \phi(\tau),
\end{align*}
$$

describe this frozen region.

When $\xi$ is positive and $O(1), (\tau - 1)$ is an $O(1)$ quantity so that the factor $\theta^2$ determines the nature of the limiting equations. In the limit the region is an equilibrium one, and

$$
\begin{align*}
\frac{d^2 \tau}{d\xi^2} &= \frac{d\tau}{d\xi} + \frac{\tilde{K}}{\theta} \phi(\tau),
\end{align*}
$$

Between these two regions there is a thin flame sheet located at $\xi = 0$ in which there is essentially a balance between the diffusion and reaction terms. The primary mathematical role of this layer is to smooth out the first derivatives which on a scale $\xi = O(1)$ are discontinuous at $\xi = 0$. The solution in the sheet has the form

$$
Y \sim 1 + \frac{1}{\theta} v_1(\eta) + \frac{1}{\theta^2} v_2(\eta) + \ldots
$$
\[
\tau \sim \tau_0 + \frac{1}{\theta} \tau_1 + \ldots
\]

where \(\eta = \theta \xi\),
and \(\tau_0\) is the maximum value of \(\tau\), (achieved as \(\eta \to \infty\), as is clear from the equation to \(t_1\), below) and

\[
\tau_0 \sim 1 + \frac{1}{\theta} \tau_0 + O\left(-\frac{1}{\theta^2}\right).
\]

The variables \(t_1\), and \(y_1\) satisfy the equations

\[
\frac{1}{\Le} \frac{d^2 y_1}{d \eta^2} = \frac{d^2 t_1}{d \eta^2} = \Omega_0 \chi(1) y_1 \exp \left[ \frac{t_1 + \tau_0}{1+\alpha^{-1}} \right]
\]

where \(\Omega_0\) is the first term in the asymptotic expansion of \(\Omega\). We make the choice

\[
y_1 = \Le t_1
\]

so that

\[
\frac{d t_1}{d \eta} \to 0, \quad \text{as} \quad \eta \to \infty.
\]

This last expression is a first integral of (5.3), (5.4) subject to the condition

\[
t_1 \to 0, \quad \frac{d t_1}{d \eta} \to 0 \quad \text{as} \quad \eta \to \infty.
\]

Of particular importance of its value as \(\eta \to -\infty\)

\[
\left[ \frac{d t_1}{d \eta} \right]_\infty^0 = 2 \Omega_0 \chi(1) \Le \int_0^{t_1} d t_1 (t_1) \exp \left[ \frac{t_1 + \tau_1}{1+\alpha^{-1}} \right].
\]

(5.5)

Solutions in the outer regions, that is the region governed by (5.1) and (5.2), have the form

\[
\tau \sim \tau_0 + \frac{1}{\theta} \tau_1 + \ldots
\]

\[
\tau_0 \sim 1 + \frac{1}{\theta} \tau_0 + \ldots
\]

Where the superscript (-) refers to the region \(\xi < 0\), (+) to the region \(\xi > 0\). General solutions are easily constructed for arbitrary functions \(\phi(\tau)\) and these must match with the flame sheet solution and satisfy the boundary conditions (2.6a) as \(\xi \to -\infty\). The condition \(\tau(\infty) = 0\) can not be invoked because of the non-uniformity when \(\xi \) is \(O(\theta)\) mentioned in the introductory remarks to this section. It is replaced by the requirement that \(\tau\) diverges at most algebraically as \(\xi \to +\infty\).

In this way we find

\[
y_0^- = \Le^{-1} e^\xi, \quad \tau_0^- = e^\xi.
\]

(5.8)

Matching of gradients with the flame-sheet solutions then implies, because of (5.7),

\[
\Omega_0 = \frac{(1 + \alpha^{-1})^2}{2 \chi(1) \Le} \exp \left[ \frac{-\tau_1}{1+\alpha^{-1}} \right]
\]

so that \(t_1\) is uniquely defined and has the asymptotic behavior

\[
\text{as} \quad \eta \to -\infty \quad t_1 \sim \eta + \mu_1^- + \text{(transc.)},
\]

(5.10)

where \(\mu_1^-\) is a known constant.

Continuing,

\[
\tau_1^- = K e^\xi \int_0^\xi d \xi' e^{-\xi'} \int_0^{\xi'} d \xi'' \phi(e^{\xi''}) + (\tau_1^- + \mu_1^-),
\]

(5.11)

\[
Y_1^- = \Le \mu_1^- e^\xi, \quad \tau_1^+ = -K \phi(1)^{\xi} + \tau_1^+.
\]

(5.12)

Further calculation, in particular the determination of \(\tau_1^+\) to complete the definition of \(\Omega_0\), requires an examination of the third-order flame-sheet structure (i.e., \(y_2, t_2\)). \(y_2\) and \(t_2\) are related through the equation
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\[
\frac{1}{\text{Le}} \frac{d^2 y_2}{dt^2} = \frac{d^2 y_1}{dt^2} = \frac{d^2 y_3}{dt^2} - \frac{d^2 t_1}{d\eta^2}, \tag{5.14}
\]

and furthermore they must have the following asymptotic behavior:

\[
\eta \to +\infty \quad y_2 \sim (\text{transc.}) \quad t_2 \sim \rho_2^+ \eta + \mu_2^+ + (\text{transc.}),
\]

\[
\eta \to -\infty \quad y_2 \sim \frac{1}{2} \text{Le}^2 \eta^2 + \alpha_2^+ \eta + \beta_2^+ (\text{transc.} \quad t_2 \sim \frac{1}{2} \eta^2 + \rho_2^- \eta + \mu_2^- + (\text{transc.}).
\]

\(\rho_2^+, \mu_2^+, \alpha_2^+ \) and \(\beta_2^+\) are unknown constants at this point, but because of (5.14) they are related:

\[\rho_2^+ = -\frac{\alpha_2^+}{\text{Le}} + \text{Le} \mu_1^- + \rho_2^- - \mu_1^- . \tag{5.15}\]

Moreover, higher-order matching with the outer solutions leads to the results

\[\rho_2^+ = -\tilde{K} \varphi (1), \]

\[\rho_2^- = \tau_1^* + \mu_1^- + \tilde{K} \int_{-\infty}^{0} \varphi (e^\xi) d \xi, \]

and

\[\alpha_2^+ = \text{Le}^2 \mu_1^- . \]

Substituting these into (5.15) determines \(\tau_1^*\),

\[\tau_1^* = -\tilde{K} \left[ \varphi (1) + \int_{-\infty}^{0} \varphi (e^\xi) d \xi \right] . \tag{5.16}\]

and then (5.9) may be written in the form

\[\frac{\omega_j}{m^2} = \frac{(1 + \alpha^{-1})^2}{2 \chi (1) \text{Le}} \exp \left[ \frac{\text{Le}^2 \tau_1^* \varphi (1) + \int_{-\infty}^{0} \varphi (e^\xi) d \xi}{(1 + \alpha^{-1})} \right]. \tag{5.17}\]

Equation (5.17) is the central result of the present paper. It reduces to the fast-flame adiabatic limit (3.2) when \(Q'\) vanishes.

When \(\omega\) is fixed, \(m\) is a multiple-valued function of \(Q'\) provided \(Q'\) is not too large, and when \(Q'\) is fixed \(m\) is a multiple-valued function of \(\omega\) provided \(\omega\) is not too small. Quenching conditions are easily deduced. When \(\omega\) is fixed

\[\frac{dm}{dQ'} \left[ \Sigma \frac{Q'}{m} - 1 \right] = \frac{\Sigma - \frac{Q'}{m}}{2m}, \]

where \(\Sigma \equiv \theta \left[ \varphi (1) + \int_{-\infty}^{0} \varphi (e^\xi) d \xi \right] (1 + \alpha^{-1}) \),

so that the quenching point (C in Fig. 1) has coordinates

\[m = \left[ 2 \omega' \chi (1) \text{Le} (1 + \alpha^{-1}) e^{-1} \right]^{1/2} , \tag{5.18a}\]

\[Q' = 2 \omega \chi (1) \text{Le} (1 + \alpha^{-1}) e^{-1} \Sigma^{-1} . \tag{5.18b}\]

Similarly, when \(Q\) is fixed the quenching point (C in Fig. 2) is

\[m = \sqrt{\Sigma Q'} , \quad \omega' = \frac{e \Sigma Q'}{2 \chi (1) \text{Le} (1 + \alpha^{-1})^2} . \tag{5.19}\]

If

\[Q' > 2 \omega' \chi (1) \text{Le} (1 + \alpha^{-1})^2 e^{-1} \Sigma^{-1} \tag{5.20}\]

there is no solution to the problem.

It is noteworthy that the flame speed at extinction is \(e^{1/2} (\approx 0.61)\) times the adiabatic fast-flame speed, a result which is independent of all physical parameters. Gerstein and Stine [3] suggested such a relationship on the basis of their numerical computations, assigning a value 0.54 to the ratio. Their results are for an activation energy of 40,000 cal/mole and an adiabatic flame temperature of 1545 °K, data consistent with large activation energy asymptotics (\(\theta \approx 13\)), so that the agreement is not fortuitous.

In addition it should be noted that the maximum temperature at extinction is only
slightly smaller than the adiabatic flame temperature. Indeed, along the whole of the fast-flame branch the difference is $O(1/0)$.

There has been some debate on the question of the effects of heat losses upstream of the flame front ($\xi < 0$) relative to losses downstream ($\xi > 0$) (e.g., Williams [10, p. 191]). This was perhaps prompted by the analysis of Spalding [7] who neglected upstream losses. The present results distinguish between the two contributions. Upstream losses make a contribution represented by the factor $\int_0^{\xi} \varphi(e^x)d\xi$, whereas downstream losses are represented by $\varphi(1)$ (Eqs. (5.16), (5.17)). These contributions are equal if $\varphi$ is proportional to $\tau$ (which is appropriate when the losses are due to conduction and $\lambda$ is a constant), but a stronger dependence on temperature (e.g., $\varphi \sim \tau^z$ representing radiation losses) diminishes the relative importance of upstream losses.

Equation (5.2) shows that for fast-flame branch, the residue of unburnt mixture behind the flame is transcendentally small in $\theta$ (cf. (4.4)). This is actually a consequence of the first order kinetic scheme. If the reaction rate was proportional to $(1 - Y)^n$, $n \neq 1$ then the residue would be algebraically small in $\theta$, for the same reason that Kassoy and Williams [5] get algebraic perturbations in their large Damkohler number study of fuel drop burning.

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References

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